Last-passage percolation on the directed complete graph on $\mathbb{Z}$ with signed weights

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Abstract

1 Introduction

We consider the model of last passage percolation on the complete graph with vertices indexed by $\mathbb{Z}$, where each edge is directed from the vertex with lower index to the vertex with larger index. Each edge carries an iid weight distributed like μ, where μ is some probability distribution on $\mathbb{R}$. Unlike many previous studies of last passage percolation, note that we allow the weights to be negative. For any finite path, we define the weight of a path to be the sum of the weights of the edges of the path. Let W_n be the maximal weight of a path from 0 to n. We take interest in the behavior of W_n as n tends to infinity. The limit $\lim_{n\to\infty} \frac{W_n}{n}$ exists a.s. by sub-additivity and is denoted by L_μ.

Question 1.1. If μ is Gumbel, then L_μ is explicit. Can we use it to conjecture universal asymptotic behaviors ?

Consider the probability distribution μ with finite support $\{a_1, \dots, a_k\}$, where $\mu(a_i) = p_i$ for every $1 \le i \le k$. One can show that L_μ is an increasing function of each a_i

Question 1.2. Can we show that L_μ is a continuous function of each a_i and each p_i ?

Let $a \in \mathbb{R} \cup \{-\infty\}$ and $0 \le p \le 1$. Define the probability measure $\mu_{p,a}$ by $\mu_{p,a}(1) = p$ and $\mu_{p,a}(a) = 1 - p$. Set $D_a(p) := L_{\mu_{p,a}}$.

Question 1.3. Compute $D_a(p)$ for any $a \in \mathbb{R} \cup \{-\infty\}$. For a fixed value of p, is the function $a \mapsto D_a(p)$ continuous in a ? Analytic on some interval ? Can one provide an interpretation of the derivative of that function ?

We expect the derivative of $a \mapsto D_a(p)$ to be given by the average number of edges of weight a in a path with maximal weight and using the maximal number of times edges of bigger weights.

When $a > 0$ the computation should be easy by considering $\lfloor a \rfloor$ if $a > 1$ or $\lfloor \frac{1}{a} \rfloor$ if $a < 1$, since one obtains a finite state space Markov chain. We also know that $D_1(p) = 1$.

Proposition 1.4. *For any* $0 \leq p \leq 1$,

$$D_0(p) = \frac{1}{\sum_{k\geq 1} (1-p)^{\frac{k(k-1)}{2}}}. \tag{1.1}$$

The function $\sum_{k\geq 1} q^{\frac{k(k-1)}{2}}$ is known as the Ramanujan ψ function.

When $a = -\infty$, we recover the Barak-Erdős graph model, thus $D_{-\infty}(p)$ equals $C(p)$, the growth rate of the longest path in the Barak-Erdős graph. When $a \leq 0$, one can try to perform a Taylor expansion around $p = 1$, by considering the missing subgraphs with an increasing number of edges. We provide more details in Section 2 for the case $a = -\infty$. For $D_0(1-q)$, the Taylor expansion as $q \to 0$ is given by an alternating sequence whose absolute values form the sequence A006950 in the OEIS. For $D_{-\infty}(1-q)$, the Taylor expansion as $q \to 0$ is given by an alternating sequence whose absolute values form the sequence A321309 in the OEIS. The sequence A006950 is dominated term-wise by the sequence A321309.

Conjecture 1.5. *For any* $a \leq 0$, *the function* $D_a(p)/p$ *is convex.*

Question 1.6. The asymptotic expansion of $D_0(p)$ as $p \to 0$ seems to be of the form

$$D_0(p) = \sqrt{p} \sum_{k\geq 0} a_k p^k$$

starting with

$$\sqrt{\frac{p}{2\pi}} \left(2 + \frac{p}{4} + \frac{19p^2}{192} + \frac{27p^3}{512} + \frac{23791p^4}{737280} + \frac{125987p^5}{5898240} + \frac{29432741p^6}{1981808640} + \frac{33805495p^7}{3170893824} + \frac{118896175213p^8}{15220290355200} + O\left(p^9\right) \right). \tag{1.2}$$

The other terms should be known, as this is the inverse of the Ramanujan ψ function, which is equal to some elliptic θ function. Can we find a probabilistic interpretation for this ?

Question 1.7. Do we also obtain integer sequences when a is a negative integer ? Do they interpolate between A006950 and A321309 ? Can one find objects counted by all these sequences ?

There are various objects counted by A006950, as explained on its OEIS webpage. However, A006950 and A321309 do not seem to count graphs with a given number of edges and some special property, as is explained in Section 2.

One can associate a particle system on $\mathbb{R}$ to the exploration of such graphs when keeping track of the maximal weight of a path starting at 0. When $a = -\infty$ we recover the infinite-bin model.

2 Perturbative expansion for Barak-Erdős graphs

2.1 Notation and main conjectures

We look at finite graphs G on k vertices labelled from 1 to k with m edges. We define $e(G) := m$ to be the order of G and $|G| := k$ to be the size of G. The set of all such graphs (for all k and m) is called $\mathcal{G}$.

We define the loss of G to be $\ell(G) := |G| + 1 - L(\widetilde{G})$, where $L(\widetilde{G})$ denotes the length of a longest path of $\widetilde{G}$, the latter being the graph on $|G| + 2$ vertices labelled from 0 to $|G|+1$ obtained by deleting from the complete graph on these $|G| + 2$ vertices the edges present in G (see Figure 2). The length of a path is measured in terms of the number of edges.

We define the multiset $S(G)$ of all subgraphs of G to be given by any subset of the $e(G)$ edges of G : there are $2^{e(G)}$ elements in $S(G)$. For example, if G is the graph on vertices $1, 2, 3$ with an edge between 1 and 2 and an edge between 2 and 3, then $S(G)$ contains : once the empty graph $G_\emptyset$ (a single vertex), twice the graph with two vertices connected by a vertex and once the full graph G. One should be careful about the vertex set of a subgraph G' of G : I think it is given by all the vertices that are adjacent to at least one edge in G'. In particular, it may be strictly smaller than the vertex set of G.

We define $\eta(G)$ recursively, using the formula

$$\ell(G) = -\sum_{G' \in S(G)} \eta(G'). \tag{2.1}$$

In particular, for the empty graph, $\ell(G_\emptyset) = \eta(G_\emptyset) = 0$.

We call a graph $G \in \mathcal{G}$ *present* if $\eta(G) \neq 0$. We denote by P_m the set of all present graphs with m edges.

Conjecture 2.1. *For any $m \geq 0$, the set P_m is finite.*

Question 2.2. It is not true that for any $G \in P_m$, $\eta(G) = (-1)^m$. This is wrong already for $m = 6$, as one sees by taking the complete graph on 4 consecutive vertices. Does η take values in $\{-1, 0, +1\}$?

Conjecture 2.3. *For q close to 0,*

$$C(1-q) = \sum_{m \geq 0} \sum_{G \in P_m} \eta(G) q^m. \tag{2.2}$$

Conjecture 2.4. *The coefficients of the above power series have alternating signs. Moreover, the sequence $(\sum_{G \in P_m} \eta(G))_{m \geq 0}$ is non-decreasing.*

Conjecture 2.3 can be rewritten as

$$C(1-q) = \sum_{g \in \mathcal{G}} \eta(G) q^{e(G)} \tag{2.3}$$

Maybe there is a relatively short proof of the fact that, whenever the series on the right-hand side of (2.3) converges, then the sum has to be $C(1-q)$, maybe using again Fubini as we did for the sum on words in our previous paper ?

It would also be nice to find the radius of convergence of this power series.

2.2 Understanding η

We now apply a Möbius inversion to the formula (2.1), to express $\eta(G)$ in terms of the length losses of the subgraphs of G.

Lemma 2.5. *Let $S_e(G)$ (resp. $S_o(G)$) be the sub-multiset of $S(G)$ consisting of subgraphs of G with an even (resp. odd) number of edges. Then for any $G \in \mathcal{G}$, we have*

$$\eta(G) = \begin{cases} \sum_{G' \in S_o} \ell(G') - \sum_{G' \in S_e} \ell(G') & \text{if } e(G) \text{ is even} \\ \sum_{G' \in S_e} \ell(G') - \sum_{G' \in S_o} \ell(G') & \text{if } e(G) \text{ is odd.} \end{cases} \tag{2.4}$$

Proof. To be written, should work by induction on the number of edges of G and using formula (2.4). □

Lemma 2.6. *If G is not connected, then $\eta(G) = 0$. In particular, if $|G| \geq e(G) + 2$, then $\eta(G) = 0$.*

Proof. We would need a clean definition of the vertex sets of G and its subgraphs G' to prove the first claim, using also Lemma 2.5 The second part follows from the fact that a connected graph with m edges has at most $m + 1$ vertices. □

So in the formula (2.1), we can restrict ourselves to connected subgraphs G' of G. Also, for a fixed order m there are finitely many graphs G with m edges such that $\eta(G) \neq 0$.

To answer Question 2.2, it would be nice to start by rewriting formula (2.4) in a way that makes clear the fact that η takes values in $\{-1, 0, 1\}$, for example by rewriting the right-hand side as the difference of two indicator functions.

2.3 A recursive construction of the present graphs

Consider the map Φ_L defined on $\mathcal{G}$ as follows. Given $G \in \mathcal{G}$, write $k = |G|$ and add to G, whose vertices are currently labelled from 1 to k, a vertex labelled 0. Add also an edge between vertex 0 and vertex 1. Then relabel all vertices from 1 to $k+1$. This new graph is defined to be $\Phi_L(G)$. We define the map Φ_R on $\mathcal{G}$, by adding to the graph G on k vertices a vertex labelled $k+1$ and an edge between vertex k and vertex $k+1$. In words, Φ_L (resp. Φ_R) adds an edge of range 1 to the left (resp. right) of a graph (where the *range* of an edge between $i < j$ is simply $j - i$).

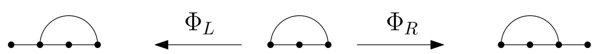

Figure 1: Example of the images of a graph under Φ_L and Φ_R.

Lemma 2.7. *For any $m \geq 0$ we have $\Phi_L(P_m) \subset P_{m+1}$ and $\Phi_R(P_m) \subset P_{m+1}$.*

Proof. Here is a rough idea of how a proof might work (no guarantee). Fix $m \geq 0$ and $G \in P_m$. One should compute $\ell(\Phi_L(G))$. Distinguish two cases. First case, there exists a longest path in $\widetilde{G}$ (the graph used to define $\ell(G)$ which does not use vertex 1 of G. Then $\ell(\Phi_G(G)) = \ell(G)$. Second case, every longest path in $\widetilde{G}$ used vertex 1 of G. Then $\ell(\Phi_L(G)) = \ell(G) + 1$. Then apply formula (2.1) to $\Phi_L(G)$, cutting the sum into two : the subgraphs that don't use the new edge added to G to make $\Phi_L(G)$ (these are subgraphs of G) and those that do. The first sum will give $\ell(G)$, since it is simply formula (2.1) applied to G. One should then deal with the second sum, and look at the two cases defined above, whether every longest path of $\widetilde{G}$ uses vertex 1 or not. □

Since the map Φ_L is clearly an injection, Lemma 2.7 implies that the sequence $(\#P_m)_{m\geq 0}$ is nondecreasing. This also gives us a way to construct many elements of P_{m+1} knowing P_m : P_{m+1} contains the sets $\Phi_L(P_m)$ and $\Phi_R(P_m)$ (which are not disjoint).

More generally, here is how I hope one could construct recursively all the present graphs. Let G be a graph on k vertices with m edges, and construct the graph $\widetilde{G}$ as in the previous section.

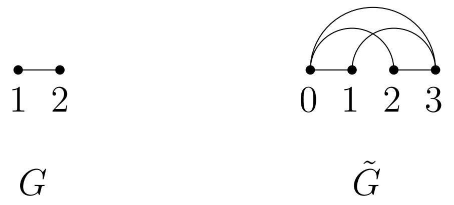

Figure 2: Example of a graph $G \in P_1$ and the corresponding graph $\widetilde{G}$.

Consider the set $\mathcal{L}(\widetilde{G})$ to be the set of all the paths of maximal length in $\widetilde{G}$ (here I should recall that the edges are oriented as in the Barak-Erdős graph, so the paths are always from left to right, I should actually have said this at the very beginning of the paper).

Figure 3: The two paths of maximal length in $\widetilde{G}$.

A collection of edges S of $\widetilde{G}$ is called a *shortening set* of $\widetilde{G}$ if the following two conditions hold :

1. any path in $\mathcal{L}(\widetilde{G})$ contains at least one edge in S ;
2. any edge in S belongs to at least one path in $\mathcal{L}(\widetilde{G})$.

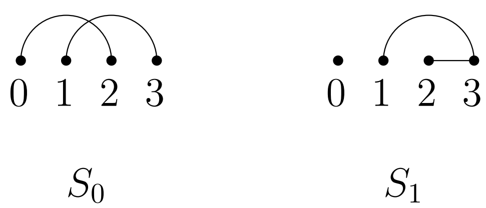

Figure 4: Two possible choices of shortening sets of $\widetilde{G}$. I could also have chosen a shortening set S_2 with 3 edges, it would maybe have been more pedagogical !

We also denote by $G + S$ the graph obtained from G by adding the edges in S, and adding vertex 0 (resp. vertex $k+1$) of $\widetilde{G}$ is S contains an edge adjacent to vertex 0 (resp. vertex $k+1$).

Conjecture 2.8. *For any $m \geq 0$ and for any $G \in P_m$, if S is a shortening set of $\widetilde{G}$ of cardinality m', then $G + S \in P_{m+m'}$.*

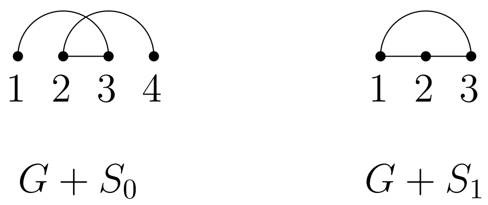

Figure 5: The two graphs $G + S_0$ and $G + S_1$, both in P_3. If I had chosen a shortening set S_2 with 3 edges, I would have obtained $G + S_2$ in P_4.

I suspect that the fact the $\Phi_L(P_m) \subset P_{m+1}$ is related to the above fact in some way. I would like to have a construction which is general enough to be able to get all graphs in P_{m+1} as coming from some smaller graphs.

References