CONDITIONS FOR ERGODICITY IN QUEUES WITH MANY SERVERS AND WAITING

S. G. Foss

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We consider here a queue with m servers and waiting. Let $w_{n,i}$ ($n \geqslant 1$ and $1 \leqslant i \leqslant m$) be the time from the month of arrival of the n-th call up to the moment of freedom of i servers from the calls that arrived before the n-th call and set $\bar{w}_n = (w_{n1}, \ldots, w_{nm})$. In this article, we investigate the conditions for the convergence of distribution of the waiting-time vector $\bar{w}_n$ as $n \to \infty$ to limit distribution.

1. Introduction

Let there be given a metrically transitive stationary sequence $\{(s_j, \tau_j); -\infty < j < \infty\}$ of random variables. Here τ_j are the intervals between the times of arrival of calls and s_j is the servicing time.

The following well-known relation is fulfilled for the vectors $\bar{w}_n$:

$$\bar{w}_{n+1} = R(\bar{w}_n + \bar{e}_1 s_n - \bar{i}\tau_n)^+, \tag{1}$$

where $\bar{e}_1 = (1, 0, \ldots, 0)$; $\bar{i} = (1, 1, \ldots, 1)$; if a is a number, then $a^+ = \max(0, a)$; if $\bar{a} = (a_1, \ldots, a_n)$, then $(\bar{a})^+ = (a_1^+, \ldots, a_n^+)$, and $R(\bar{a})$ is the rearrangement of the components of the vector $\bar{a}$ in decreasing order.

We will assume that the mathematical expectations $M\{s_1\}$ and $M\{\tau_1\}$ exist, are finite, and satisfy the inequality

$$M\{s_1\} - mM\{\tau_1\} < 0. \tag{2}$$

We are interested in the conditions for the weak convergence of the distributions of the random vectors $\{\bar{w}_n;\ n = 0, 1, \ldots\}$ to a certain limit distribution.

We cite two well-known results.

THEOREM 1 [4]. If $\{s_j\}$ and $\{\tau_j\}$ are two independent sequences of identically distributed independent random variables that satisfy (2), then there exists a proper stationary sequence $\{\bar{w}^n\}$ that satisfies (1) and is such that the distribution of $\bar{w}_n$ converges weakly to the distribution of the vector $\bar{w}^0$ as $n \to \infty$ for arbitrary initial condition $\bar{w}_0$.

THEOREM 2 [1, 5]. If $\bar{w}_0 = 0$, then there exists a proper stationary sequence $\{\bar{w}^n\}$ that satisfies (1) and is such that the distribution function of $\bar{w}_n$ converges monotonically to the distribution function of the vector $\bar{w}^0$.

Let us observe that if $\bar{w}_0 \neq 0$, then the distribution function of $\bar{w}_n$ does not necessarily converge to any limit distribution function for an arbitrary metrically transitive stationary sequence $\{(s_j, \tau_j)\}$. An appropriate example is given in [5, p. 516].

Let $\mathscr{F}_{n,l}$ be the σ-algebra generated by the variables $(s_n, \tau_n), \ldots, (s_{n+l}, \tau_{n+l})$. Let T denote a one-to-one measure-preserving transformation shift of sets from the σ-algebra $\mathscr{F} = \mathscr{F}_{-\infty,\infty}$ such that

$$T\{\omega : (s_j, \tau_j) \in B_j;\ j = 1, \ldots, k\} = \{\omega : (s_{j+1}, \tau_{j+1}) \in B_j,\ j = 1, \ldots, k\}$$

for each family of two-dimensional Borel sets B_j. Let us denote the corresponding transformations over $\mathscr{F}$ -measurable functions by U, so that $(s_{j+1}, \tau_{j+1}) = U(s_j, \tau_j)$. The transformations T^{-1} and U^{-1}, and, by the same token, T^n and U^n for arbitrary integer n, are defined in the same manner (here T^0 and U^0 are the identity transformations). We will suppose that the σ-algebra $\mathscr{F}$ is nondegenerate (i.e., at least one of the variable s_0 and τ_0 is nondegenerate) and all the considered random variables are given on a single probability space and are measurable with respect to $\mathscr{F}$.

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It follows from Theorem 2 that there always exists a stationary sequence $\{\overline{w}^n = U^n\overline{w}^0;\ -\infty < n < \infty\}$ that satisfies (1). *The condition A is said to be fulfilled* if this sequence is unique. In other words, the condition A means that if $\{\overline{a}^n = U^n\overline{a}^0;\ -\infty < n < \infty\}$ is a stationary sequence that satisfies (1), then $\overline{a}^n = \overline{w}^n$ almost surely (a.s.).

In the present article, we prove the following theorem.

THEOREM 3. The distribution function of the vector $\overline{w}_n$ converges to a limit distribution function as $n \to \infty$ for an arbitrary initial condition $\overline{w}_0$ if and only if the condition A is fulfilled.

The following theorem is useful in finding sufficient conditions for the fulfillment of A.

THEOREM 4. Let $\{\overline{a}^n = U^n\overline{a}^0;\ -\infty < n < \infty\}$ be a stationary sequence that satisfies (1) and is distinct from $\{\overline{w}^n\}$ (i.e., $P\{\overline{a}^0 = \overline{w}^0\} < 1$). Then the following statements are valid:

a) $P\{\overline{a}^0 \geqslant \overline{w}^0\} = 1$;

b) if $\overline{a}^0 = (a_1^0, \ldots, a_m^0)$ and $\overline{w}^0 = (w_1^0, \ldots, w_m^0)$, then $\sum_{i=1}^{m}(a_i^0 - w_i^0) = \text{const} > 0$ a.s. and, consequently, $P\{\overline{a}^0 = \overline{w}^0\} = 0$;

c) $(a_1^0 + s_0 - \tau_0)^+ - (w_1^0 + s_0 - \tau_0)^+ = a_1^0 - w_1^0$ a.s. and $(a_i^0 - \tau_0)^+ - (w_i^0 - \tau_0)^+ = a_i^0 - w_i^0 n$ a.s. for $i = 2, \ldots, m$.

Let us now consider a somewhat different problem — to find the conditions for the so-called "strong" convergence of the distributions of the sequence $\{\overline{w}_n\}$.

Definition. A sequence of random vectors $\{\overline{w}_n\}$ is said to *converge strongly to a random vector* $\overline{w}^0$ *as* $n \to \infty$ if

$$P\{U^{-k}\overline{w}_k = \overline{w}^0 \quad \text{for} \quad k \geqslant n\} \to 1 \tag{3}$$

as $n \to \infty$.

To find the conditions for strong convergence, Borovkov [3] proposed the method of renewal events. Let us recall its definition.

An event $A_n \in \mathcal{F}_{-\infty, n+L}$ is said to be *renewal* on the interval $[n, n + L]$ if for $k > L$ the random vectors $\overline{w}_{n+k} = \overline{w}_{n+k}(\omega)$ on the set $\omega \in A_n$ admit the representation

$$\overline{w}_{n+k} = \varphi(s_n, \tau_n, \ldots, s_{n+k-1}, \tau_{n+k-1}),$$

where the form of the function φ depends only on the number of arguments and is determined by the choice of the sequence A_n.

THEOREM 5 [2, 3]. Let there exist a sequence of the renewal events A_j such that

$$P\left\{\bigcap_{l=l_0}^{\infty} \bigcup_{j=1}^{n} A_j T^{-l} A_{j+l}\right\} \to 1 \tag{4}$$

as $n \to \infty$ for some $l_0 > 0$. Then the sequence $\{\overline{w}_n\}$ converges strongly to a random vector $\overline{w}^0$ as $n \to \infty$; in addition, the stationary sequence $\{\overline{w}^n = U^n\overline{w}^0\}$ satisfies (1). If the sequence $\{A_j\}$ is stationary (i.e., $A_{j+1} = TA_j$), then the condition $P\{A_0\} > 0$ is sufficient for the fulfillment of (4).

In this article, we prove the following theorem (converse of Theorem 5).

THEOREM 6. Let a sequence of random vectors $\{\overline{w}_n\}$ converge strongly to a random vector $\overline{w}^0$ as $n \to \infty$. Then there exists a sequence $\{A_n\}$ of renewal events that satisfies (4).

Let us now consider renewal events of the form

$$A_n = \{f_1(\overline{w}_n, \ldots, \overline{w}_{n+L}, s_n, \ldots, s_{n+L}, \tau_n, \ldots, \tau_{n+L}) \leqslant 0, \ldots, f_k(\overline{w}_n, \ldots, \overline{w}_{n+L}, s_n, \ldots, s_{n+L}, \tau_n, \ldots, \tau_{n+L}) \leqslant 0\}, \tag{5}$$

where the functions $f_1, \ldots, f_k$ are nondecreasing with respect to the arguments corresponding to $\overline{w}_i$ and s_i and nonincreasing with respect to the arguments corresponding to τ_i.

Let ε be a positive number, $\overline{w}_0^{(\varepsilon)} = 0$, and $\overline{w}_{n+1}^{(\varepsilon)} = R(\overline{w}_n^{(\varepsilon)} + \overline{e}_1(s_n + \varepsilon) - \overline{i}\tau_n)^+$ for $n = 0, 1, \ldots$.

The following theorem is valid.

THEOREM 7. For a certain $\varepsilon > 0$ and for a sequence $\overline{w}_n^{(\varepsilon)}$ let there exist renewal events $\{A_n^{(\varepsilon)}\}$ of the form (5) such that

$$P\left\{\bigcup_{n=l}^{\infty} A_n^{(\varepsilon)}\right\} = 1$$

for $l = 1, 2, \ldots$. Then the assertion of Theorem 5 is valid for the sequence $\{\overline{w}_n\}$ for arbitrary initial condition $\overline{w}_0$.

We give one of the corollaries that follow from Theorem 7.

COROLLARY 1. Let the random vectors (s_j, τ_j) be independent for different j. Then the assertion of Theorem 5 is valid for the sequence $\{\overline{w}_n\}$ for arbitrary initial condition $\overline{w}_0$.

Corollary 1 is a strengthening of Theorem 6 of [2].

2. Auxiliary Results

LEMMA 1. Let $\bar{a}_0 = (a_{01}, \ldots, a_{0m})$ and $\bar{b}_0 = (b_{01}, \ldots, b_{0m})$ (where $a_{01} \leqslant \ldots \leqslant a_{0m}$ and $b_{01} \leqslant \ldots \leqslant b_{0m}$) be two m-dimensional vectors and c, d, and τ be nonnegative numbers. If $\bar{b}_1 = R(\bar{b}_0 + \bar{e}_1 c - \bar{i} d)^+$ and $\bar{a}_1 = R(\bar{a}_0 + \bar{e}_1(c+\varepsilon) - \bar{i} d)^+$, then

$$\sum_{i=1}^{m} (b_{1i} - a_{1i})^+ \leqslant \sum_{i=2}^{m} (b_{0i} - a_{0i})^+ + (b_{01} - a_{01} - \varepsilon)^+.$$

Proof. The proof of this lemma is based on the following well-known inequalities:

1) $(x^+ - y^+)^+ \leqslant (x - y)^+$ for arbitrary numbers x and y.

2) If $x_1 \leqslant \ldots \leqslant x_m$ and $y_1 \leqslant \ldots \leqslant y_m$, then

a) $(y_1 - x_1)^+ + \ldots + (y_m - x_m)^+ \leqslant (y_1 - x_m)^+ + (y_2 - x_1)^+ + \ldots + (y_m - x_{m-1})^+$,

b) $(y_1 - x_1)^+ + \ldots + (y_m - x_m)^+ \leqslant (y_m - x_1)^+ + (y_1 - x_2)^+ + \ldots + (y_{m-1} - x_m)^+$.

Now, let k be such that $(b_{0k} - d)^+ \leqslant (b_{01} + c - d)^+ < (b_{0,k+1} - d)^+$, and l be such that $(a_{0l} - d)^+ \leqslant (a_{01} + c + \varepsilon - d)^+ < (a_{0,l+1} - d)^+$. Then

$$b_{1i} = \begin{cases} (b_{0,i+1} - d)^+ & \text{for } i < k, \\ (b_{01} + c - d)^+ & \text{for } i = k, \\ (b_{0i} - d)^+ & \text{for } i > k, \end{cases}$$

$$b_{1i} = \begin{cases} (a_{0,i+1} - d)^+ & \text{for } i < l, \\ (a_{01} + c + \varepsilon - d)^+ & \text{for } i = l, \\ (a_{0i} - d)^+ & \text{for } i > l, \end{cases}$$

Let us consider three cases: α) $k = l$; β) $k < l$; γ) $k > l$.

Case α). The following relations are valid:

$$\sum_{i=1}^{m} (b_{1i} - a_{1i})^+ = \sum_{i=1}^{k-1} \left((b_{0,i+1} - d)^+ - (a_{0,i+1} - d)^+\right)^+ + \left((b_{01} + c - d)^+ - \right.$$

$$\left. - (a_{01} + c + \varepsilon - d)^+\right)^+ + \sum_{i=k+1}^{m} \left((b_{0i} - d)^+ - (a_{0i} - d)^+\right)^+ \leqslant \sum_{i=2}^{m} (b_{0i} - a_{0i})^+ + (b_{01} - a_{01} - \varepsilon)^+.$$

Case β). The following relations are valid:

$$\sum_{i=1}^{m} (b_{1i} - a_{1i})^+ = \sum_{i=1}^{k-1} + \sum_{i=k}^{l} + \sum_{i=l+1}^{m} \leqslant \text{(by the inequality a))}$$

$$\leqslant \sum_{i=1}^{k-1} + (b_{1k} - a_{1l})^+ + \sum_{i=k+1}^{l} (b_{1i} - b_{1,j-1})^+ + \sum_{i=l+1}^{m} \leqslant \sum_{i=2}^{m} (b_{0i} - a_{0i})^+ + (b_{01} - a_{01} - \varepsilon)^+.$$

Case γ). This case is considered in the same manner as the case β) with the application of the inequality b). The lemma is proved.

Let there be given two sequences of nonnegative numbers $\{c_i\}_{i=0}^{\infty}$ and $\{d_i\}_{i=0}^{\infty}$ and also a vector $\bar{x} = (x_1,\ldots,x_m)$. Let us set $h(x) = \min\{k : x_k = \min x_i\}$.

We introduce the following sequences of vectors: For $k = 0, 1, \ldots$ we set

$$\bar{b}_{k+1} = R(\bar{b}_k + \bar{e}_1 c_k - \bar{i} d_k)^+;\quad \bar{a}_{k+1} = R(\bar{a}_k + \bar{e}_1(c_k + \varepsilon) - \bar{i} d_k)^+;$$

$$\bar{b}'_{k+1} = \left(\bar{b}'_k + \bar{e}_{h(\bar{b}'_k)} C_k - \bar{i} d_k\right)^+;$$

$$\bar{a}'_{k+1} = \left(\bar{a}'_k + \bar{e}_{h(\bar{a}'_k)}(c_k + \varepsilon) - \bar{i} d_k\right)^+;$$

where $\bar{b}'_0 = \bar{b}_0$ and $\bar{a}'_0 = \bar{a}_0$. Let us observe that the following equalities are always fulfilled: $\bar{b}_k = R(\bar{b}'_k)$ and $\bar{a}_k = R(\bar{a}'_k)$. Let us set $\nu_0 = \min\{k \geqslant 1 : b_{l1} \leqslant a_{l1}$ and $b_{lm} \leqslant a_{lm}$ for l $k\}$ and $\nu_0 = \infty$ if no such k exists.

LEMMA 2. If at least one of the conditions

$$\sum c_i = \infty \quad \text{or} \quad \sum d_i = \infty$$

is fulfilled, then $\nu_0 < \infty$.

Proof. We introduce the following notation:

$$\nu_1(r) = \min\{k \geqslant 1: \text{ for } l \geqslant k\ \ b'_{l,r} \leqslant a_{l,m}\},$$
$$\nu_2(r) = \min\{k \geqslant 1: \text{ for } l \geqslant k\ \ b_{l,1} \leqslant a'_{l,r}\}$$

for $r = 1,\ldots,m$. Then $\nu_0 = \max\limits_{1 \leqslant r \leqslant m} \max\{\nu_1(r), \nu_2(r)\}$. We show that $\nu_1(r)$ is finite for $1 \leqslant r \leqslant m$.

Let $k_1 < k_2 < \ldots < k_n < \ldots$ be numbers such that $h(\bar{b}'_{k_n}) = r$. Let us observe that if $\sum d_i < \infty$ and $\sum c_i = \infty$, then there are always infinitely many such numbers. But if $\sum d_i = \infty$, then in the case where there are only finitely many numbers k_n there exists a number n_0 such that $\bar{b}'_{l,r} = 0$ for $l \geqslant n_0$. Therefore $\nu_1(r) \leqslant n_0 < \infty$.

Let $\{k_n\}$ be an infinite sequence. Then, by virtue of Lemma 1, the event $\{b'_{k_n,r} \geqslant a_{k_n,1} + \varepsilon\}$ can happen at most $m \cdot ([a_{0m}/\varepsilon] + 1)$ times (here $[x]$ is the integral part of x). Consequently, there exists a number n_0 such that $b'_{k_n,r} < a_{k_n,r} + \varepsilon$ for $n \geqslant n_0$. Hence for each $n \geqslant n_0$ there exists a $j_n \in \{1, \ldots, m\}$ such that $b'_{i,r} \leqslant a'_{i,j_n}$ for $k_n + 1 \leqslant i \leqslant k_{n+1}$. Since $a_{i,m} = \max\limits_{l} a_{i,l}$, it follows that $b'_{i,r} \leqslant a_{i,m}$ for $i \geqslant k_{n_0} + 1$. By the same token, $\nu_1(r) \leqslant k_{n_0} + 1 < \infty$. The finiteness of $\nu_1(r)$ is proved.

We now prove the finiteness of $\nu_2(r)$. Let $k_1 < \ldots < k_n < \ldots$ be such that $h(\bar{a}'_{k_n}) = r$. If the sequence $\{k_n\}$ is infinite, then there exists an n_0 such that $b_{k_n,1} < a'_{k_n,r} + \varepsilon$ for $n \geqslant n_0$. Consequently, $\nu_2(r) \leqslant k_{n_0} + 1 < \infty$. Let the sequence $\{k_n\}$ be finite. Let $1 \leqslant r_1 < \ldots < r_j \leqslant m$ denote all the numbers such that the sequences $\{k_n(r_1)\}, \ldots, \{k_n(r_j)\}$ are infinite. Then there exists an $i_0 < \infty$ such that $a_{i1} = \min\limits_{1 \leqslant l \leqslant j} a'_{i,r_l}$ for $i \geqslant i_0$. Set $n_0 = \max\{i_0, \nu_2(r_j), \ldots, \nu_2(r_j)\}$. Then $a'_{n,r_l} \geqslant b_{n1}$ for $n \geqslant n_0$, $1 \leqslant l \leqslant j$, and, consequently, $a'_{n,r} \geqslant a_{n1} \geqslant b_{n1}$. By the same token, $\nu_2(r) \leqslant n_0 < \infty$. The lemma is proved.

Remark 1. If the sequence $\{(\tau_j, s_j)\}$ is metrically transitive, then $P\{\sum \tau_i = \infty\} = 1$ provided $P\{\tau_0 > 0\} > 0$. By virtue of the condition (2), this inequality is always fulfilled.

Remark 2. Let $\bar{x} = (x_1,\ldots,x_m)$ and $\bar{y} = (y_1,\ldots,y_m)$ be two vectors. If $x_1 \leqslant y_1, \ldots, x_m \leqslant y_m$, then we will use the expression $\bar{x} \leqslant \bar{y}$. Therefore, if $\bar{x} \leqslant \bar{y}$, then $R(\bar{x}) \leqslant R(\bar{y})$, $(\bar{x})^+ \leqslant (\bar{y})^+$, $\bar{x} + \bar{e}_1 c \leqslant \bar{y} + \bar{e}_1 c$, and $\bar{x} - \bar{i} d \leqslant \bar{y} - \bar{i} d$ for arbitrary numbers $c \geqslant 0$ and d.

LEMMA 3. If at least one of the two conditions $\sum c_i = \infty$ and $\sum d_i = \infty$ is fulfilled, then

$$\nu^0 = \min\{k \geqslant 1: \text{ for } l \geqslant k\ \bar{b}_l \leqslant \bar{a}_l\} < \infty.$$

Proof. Let the conditions of the lemma be fulfilled. Then $\nu_0 < \infty$. For $k = \nu_0, \nu_0 + 1, \ldots$ we set

$$u(k) = \min\{l : 1 \leqslant l \leqslant m \text{ and for } l \leqslant i \leqslant m \ b_{ki} \leqslant a_{ki}\}.$$

Let us observe that $u(k) \leqslant m$ and $u(k) \neq 2$ for $k \geqslant \nu_0$. We must prove that there exists a $k \geqslant \nu_0$ such that $u(k) = 1$.

Let us observe that the sequence $\{u(k);\ k \geqslant \nu_0\}$ is nonincreasing. Indeed, we take an arbitrary $k \geqslant \nu_0$. If $u(k) = 1$, then we obviously have $u(k + 1) = 1$. Let $u(k) = l \geqslant 3$. Set

$$r(a, k) = \min\{n : (a_{k1} + s_k + \varepsilon - \tau_k)^+ \leqslant (a_{k,\,n+1} - \tau_k)^+\}$$

and $r(a, k) = m$ if no such n exists; we define $r(b, k)$ analogously. Therefore if $\max(r(a, k), r(b, k)) < u(k)$, then $u(k + 1) \leqslant u(k)$. But if $\max(r(a, k), r(b, k)) \geqslant u(k)$, then, as is easily verified, $u(k + 1) \leqslant u(k) - 1$.

Since the sequence $\{u(k)\}$ is integral, positive, and nonincreasing, the limit $\lim u(k) = u$ exists and, in addition, $u(k) = u$, starting from a certain number $k_0 < \infty$. Let $u \neq 1$. Then $\max(r(a, k), r(b, k)) < u$ for $k \geqslant k_0$. This is possible only when $\sum d_i = \infty$. But then $b_{k+1,\,m} = (b_{km} - d_k)^+$ and $a_{k+1,\,m} = (a_{km} - d_k)^+$ for $k \geqslant k_0$. Consequently, $b_{km} = a_{km} = 0$, and therefore, $\bar{b}_k = \bar{a}_k = 0$, starting from a certain number k_1. We have obtained a contradiction. By the same token, the lemma is proved.

We introduce some notation. Let $\bar{u}_n^0 = 0$ for $-\infty < n < \infty$ and

$$\bar{u}_{n+1}^{k+1} = R(\bar{u}_n^k + \bar{e}_1(s_n + \varepsilon) - \bar{i}\tau_n)^+$$

for $k \geqslant 0$ and $\varepsilon > 0$; $\bar{v}_n^0 = \bar{b} = (b_1, \ldots, b_m)$ (where $b_1 \leqslant \ldots \leqslant b_m$) ; and $\bar{v}_{n+1}^{k+1} = R(\bar{v}_n^k + \bar{e}_1 s_n - \bar{i}\tau_n)^+$ for $k \geqslant 0$.

Let $\nu^{-k} = \min\{r \geqslant 1 : \bar{b}_r \leqslant \bar{a}_r\}$ for $\bar{a}_0 = \bar{u}_0^k$, $\bar{b}_0 = \bar{v}_0^k$, $c_l = s_l$, and $d_l = \tau_l$ and $\nu = \max\limits_{k \geqslant 0} \nu^{-k}$.

LEMMA 4. If (2) is fulfilled, then $P\{\nu < \infty\} = 1$.

Proof. It follows from Theorem 2 that $\bar{u}_0^k$ do not decrease as k increases and converge to a finite limit $\bar{w}^0$ a.s. By virtue of Lemma 1, for arbitrary $k \geqslant 0$ we have

$$\sum_{l=1}^{m} (v_{0l}^k - w_l^0)^+ \leqslant \sum_{l=1}^{m} (v_{0l}^k - u_{0l}^k)^+ \leqslant \sum_{l=1}^{m} b_l.$$

Then, by virtue of Remark 2, $\bar{v}_0^k \leqslant \bar{w}^0 + \bar{i}\left(\sum b_l\right)$ a.s.

For $\bar{a}_0 = 0$ and $\bar{b}_0 = \bar{w}^0 + \bar{i}\left(\sum b_l\right)$, we set

$$\nu^* = \min\{r \geqslant 1 : \bar{b}_r \leqslant \bar{a}_r\}.$$

Then, as is easily seen, $\nu^* \geqslant \nu^{(-k)}$ a.s. for each $k \geqslant 0$ and $\nu^* \geqslant \nu$ a.s. By virtue of Lemma 3, the random variable ν^* is finite a.s. Consequently, ν is finite. The lemma is proved.

LEMMA 5. Let x be a positive number, $\bar{b}_0 = \bar{w}^0 + \bar{i}x$, and $\bar{b}_n = R(\bar{b}_{n-1} + \bar{e}_1 s_{n-1} - \bar{i}\tau_{n-1})^+$ for $n = 1, 2, \ldots$. Then there exists a stationary sequence $\{\bar{b}^n = U^n \bar{b}^0;\ -\infty < n < \infty\}$ that satisfies (1) and is such that the distribution of $\bar{b}_n$ converges to the distribution of b^0 as $n \to \infty$.

Proof. Let us set $\bar{c}_n^k = U^{-n+k}\bar{b}_n$, $\bar{c}_0^0 = \bar{b}_0$. We observe that

$$\bar{b}_1 = R(\bar{b}_0 + \bar{e}_1 s_0 - \bar{i}\tau_0)^+ = R(\bar{w}^0 + \bar{i}x + \bar{e}_1 s_0 - \bar{i}\tau_0)^+ \leqslant$$
$$\leqslant R(\bar{w}^0 + \bar{e}_1 s_0 - \bar{i}\tau_0)^+ + \bar{i}x = \bar{w}^1 + \bar{i}x = U(\bar{b}^0) \text{ a.s.}$$

Consequently, $U^{-1}(\bar{b}_1) \leqslant \bar{b}_0$ a.s., i.e., $\bar{c}_1^0 \leqslant \bar{c}_0^0$ a.s. By virtue of monotonicity, $P\{\bar{b}_2 \leqslant R(\bar{w}^1 + \bar{i}x + \bar{e}_1 s_1 - \bar{i}\tau_1)^+\} = 1$. Consequently, $P\{\bar{c}_2^0 \leqslant \bar{c}_1^0\} = 1$. Analogously, $P\{\bar{c}_{n+1}^0 \leqslant \bar{c}_n^0\} = 1$ for $n = 2, 3, \ldots$. Since $P\{\bar{c}_n^0 \geqslant \bar{w}^0\} = 1$ for $n = 0, 1, \ldots$, the limit $\lim\limits_{n\to\infty} \bar{c}_n^0 = \bar{b}^0 \geqslant \bar{w}^0$ exists a.s. In exactly the same manner, $\lim\limits_{n\to\infty} \bar{c}_n^k = \bar{b}^k \geqslant \bar{w}^k$ exists a.s. for $k = 1, 2, \ldots$. Let us observe that $R(\bar{c}_n^k + \bar{e}_1 s_k - \bar{i}\tau)^+ = \bar{c}_{n+1}^{k+1}$; $\bar{b}^{k+1} = \lim \bar{c}_n^{k+1} = \lim U(\bar{c}_n^k) = U(\bar{b}^k)$, and $\bar{b}^{k+1} = R(\bar{b}^k + \bar{e}_1 s_k - \bar{i}\tau_k)^+$, which was desired to be proved.

3. Proof of Theorems 3, 4, 6, and 7

Proof of Theorem 6. Let $\overline{w}^{k+1}=U^{k+1}\overline{w}^0$ for $k \geqslant 0$ and $n \geqslant 1$. Set $L=\min\{l\geqslant 1 : P\{U^{-l}\overline{w}_l=\overline{w}^0\}>0\}$, $C_0=\{\overline{w}_L=\overline{w}^L\}$, $C_k=T^kC_0$, and $B_k=\{\overline{w}_k=\overline{w}^k\}$ for $k = 1, 2, \ldots$.

Let us observe that the renewal takes place on the event $A_k = C_kB_{k+L}$ for the sequence $\{\overline{w}_n\}$ on the interval $[k, k + L - 1]$.

We show that the events $\{A_n\}$ satisfy the condition (4). Indeed, if we set

$$\lambda=\min\{n : U^{-k}\overline{w}_k=\overline{w}^0;\ k=n,\ n+1, \ldots\},$$

then

$$\bigcap_{l=1}^{\infty}\bigcup_{j=1}^{n} A_jT^{-l}A_{j+l}\supseteq\bigcap_{j=1}^{n} A_j\bigcap_{l=1}^{\infty} T^{-l}A_{j+l}=\bigcup_{j=1}^{n} C_jB_{j+L}\bigcap_{l=1}^{\infty} T^{-l}B_{j+L+l}$$
$$=\bigcup_{j=1}^{n} C_jT^{j+L}(\{\lambda\leqslant j+L\})=\bigcup_{j=1}^{n} T^j\left(C_0T^L(\{\lambda\leqslant j+L\}\right)=H.$$

Let $k=\min\{j : P\{C_0T^L(\{\lambda\leqslant j+L\})\}>0\}$. Let us set $D_0=C_0T^L(\{\lambda\leqslant k+L\})$ and $D_l = T^lD_0$ for $l = 1, 2, \ldots$. Then

$$H\supseteq\bigcup_{j=k}^{n} T^j\left(C_0T^L(\{\lambda\leqslant j+L\})\right)\supseteq\bigcup_{j=k}^{n} D_j.$$

Since $P\{D_0\} > 0$ and $D_j = T^jD_0$, it follows that $P\left\{\bigcup_{j=k}^{n} D_j\right\}\to 1$ as $n \to \infty$, which was desired to be proved.

Remark 3. The proof of Theorem 6 is valid (without any changes) under the more general conditions of [3] also.

Proof of Theorem 7. Let the conditions of the theorem be fulfilled. Since the events A_n and $A_n^{(\varepsilon)}$ have the form (5), we have

$$T^{-l}A_{j+l}\cap\{\nu\leqslant n\}\supseteq A_j^{(\varepsilon)}\cap\{\nu\leqslant n\},$$

for $j \geqslant n$ and $l \geqslant 0$. Let us set

$$B_n=\bigcup_{j=n}^{\infty} A_j\bigcap_{l=1}^{\infty} T^{-l}A_{j+l}\quad\text{for}\quad n=1, 2, \ldots.$$

We take an arbitrary $\delta > 0$ and find an $n=n(\delta)=\min\{k : P\{\nu>k\}\leqslant\delta\}$, depending on it. Then

$$P\{B_1\}\geqslant P\{B_n\}\geqslant P\{B_n\cap\{\nu\leqslant n\}\}=$$
$$=P\left\{\bigcup_{j=n}^{\infty}\left(\{A_j\cap\{\nu\leqslant n\}\}\bigcap_{l=1}^{\infty}\{T^{-l}A_{j+l}\cap\{\nu\leqslant n\}\}\right)\right\}\geqslant P\left\{\bigcup_{j=n}^{\infty}\{A_j^{(\varepsilon)}\cap\{\nu\leqslant n\}\}\right\}\geqslant 1-P\{\nu>n\}\geqslant 1-\delta.$$

By virtue of arbitrariness of $\delta > 0$, Theorem 7 is proved.

Proof of Corollary 1. We will consider renewal events of the form

$$A_n=\left\{w_{n+j,1}=0;\ j=0, \ldots, m-1;\ w_{n,m}\leqslant\sum_{j=0}^{m-1}\tau_{j+n}\right\}.$$

Since $M\{s_j - m\tau_j\} < 0$, there exists an $\varepsilon > 0$ such that $M\{s_j - m\tau_j\} < -\varepsilon$. Let us set $s_j' = s_j + \varepsilon$. Then $M\{s_j' - m\tau_j\} < 0$.

By the same token, it is sufficient for us to prove the following statement: If the conditions of Corollary 1 are fulfilled, then under the initial conditions $\overline{w}_0 = 0$ there exist stationary renewal events $\{A_n\}$ such that $P\{A_0\} > 0$.

By virtue of (2), $P\{m\tau_j - s_j > 0\} > 0$. Therefore, there exists a $\delta > 0$ such that $P\{m\tau_j - s_j > \delta\} > 0$. Since

$$\{m\tau_j-s_j>\delta\}\subseteq\bigcup_{a_l\in I}\{\tau_j\geqslant a_l;\ s_j\leqslant ma_l-\delta\},$$

where I is the set of positive rational numbers, there exists an $\alpha > 0$ such that $P\{\tau_j \geqslant \alpha; s_j \leqslant m\alpha - \delta\} > 0$. By virtue of Theorem 2, $\bar{w}_0$ is a proper random vector. Consequently, there exists a number $x < \infty$ such that $P\{w_m^0 < x\} > 0$. Since $\bar{w}^0$ does not depend on $\{\tau_j, s_j); j \geqslant 1\}$, for arbitrary number $N \geqslant 0$ we have

$$P\{w_m^0 < x;\ \tau_j \geqslant \alpha;\ s_j \leqslant m\alpha - \delta;\ j = 0, \ldots, N\} = P\{w_m^0 < x\} \cdot \prod_{j=0}^{N} P\{\tau_j \geqslant \alpha;\ s_j \leqslant m\alpha - \delta\} > 0.$$

The remaining part of the proof of Corollary 1 coincides verbatim with the proof of Theorem 7 of [3].

Proof of Theorem 4. For a vector $\bar{x} = (x_1, \ldots, x_m)$ we set $\|\bar{x}\| = x_1^+ + \ldots + x_m^+$.

We prove a). Let us observe that $\bar{a}^0 \geqslant \bar{w}_0 = 0$ a.s. Consequently, $\bar{a}^n \geqslant \bar{w}_n$ a.s. for $n = 1, 2, \ldots$. By virtue of Theorem 2, $\bar{w}_n \Rightarrow \bar{w}^0$ as $n \to \infty$ and $P\{\|\bar{w}^n - \bar{w}_n\| > \varepsilon\} \to 0$ as $n \to \infty$. Since $\|\bar{w}^n - \bar{a}^n\| \leqslant \|\bar{w}^n - \bar{w}_n\|$, it follows that $P\{\|\bar{w}^n - \bar{a}^n\| > \varepsilon\} \to 0$ as $n \to \infty$. But $\|\bar{w}^n - \bar{a}^n\| = U^n\|\bar{w}^0 - \bar{a}^0\|$. Consequently, $P\{\|\bar{w}^n - \bar{a}^n\| > \varepsilon\} = P\{\|\bar{w}^0 - \bar{a}^0\| > \varepsilon\}$. Therefore, $P\{\|\bar{w}^0 - \bar{a}^0\| > \varepsilon\} = 0$ for arbitrary $\varepsilon > 0$, which was desired to be proved.

We prove b). By virtue of Lemma 1, $\|\bar{a}^{n+1} - \bar{w}^{n+1}\| = \|R(\bar{a}^n + \bar{e}_1 s_n - \bar{i}\tau_n)^+ - R(\bar{w}^n + \bar{e}_1 s_n - \bar{i}\tau_n)^+\| \leqslant \|\bar{a}^n - \bar{w}^n\|$ a.s. But since $\|\bar{a}^{n+1} - \bar{w}^{n+1}\| = U(\bar{a}^n - \bar{w}^n\|)$, it follows that $\|\bar{a}^n - \bar{w}^n\| = \|\bar{a}^{n+1} - \bar{w}^{n+1}\|$ a.s. By virtue of the metric transitivity of the sequence $\{(s_j, \tau_j)\}$, the random variable $\|\bar{a}^0 - \bar{w}^0\|$ must be degenerate.

We prove c). Let us observe that

$$\|\bar{a}^1 - \bar{w}^1\| = (a_1^0 + s_0 - \tau_0)^+ - (w_1^0 + s_0 - \tau_0)^+$$
$$+ \sum_{k=2}^{m} [(a_k^0 - \tau_0)^+ - (w_k^0 - \tau_0)^+] = \|\bar{a}^0 - \bar{w}^0\| = (a_1^0 - w_1^0) + \sum_{k=2}^{m} (a_k^0 - w_k^0).$$

Since $(a + x)^+ - (b + x)^+ \leqslant a - b$ for arbitrary numbers a, b, and x such that $a \geqslant b$, we have

$$(a_1^0 + s_0 - \tau_0)^+ - (w_1^0 + s_0 - \tau_0)^+ = a_1^0 - w_1^0 \text{ a.s. and } (a_k^0 - \tau_0)^+$$
$$- (w_k^0 - \tau_0)^+ = a_k^0 - w_k^0 \text{ a.s. for } k = 2, \ldots, m.$$

Proof of Theorem 3. Necessity. Let the condition A be not fulfilled. Then there exists a stationary sequence $\{\bar{a}^n = U^n\bar{a}^0\}$ that satisfies (1) and is such that $P\{\bar{a}^0 = \bar{w}^0\} < 1$. By virtue of Theorem 4, $P\{\bar{a}^0 = \bar{w}^0\} = 0$. Since we have assumed that at least one of random variables s_0 and τ_0 is nondegenerate, there exists a set $C \in \sigma\{(s_0, \tau_0)\}$ such that $0 < P\{C\} < 1$. Let us set $\bar{b}_0 = \bar{a}^0 \cdot I\{C\} + \bar{w}^0(1 - I\{C\})$ and $\bar{b}_{n+1} = R(\bar{b}_n + \bar{e}_1 s_n - \bar{i}\tau_n)^+$ for $n = 0, 1, \ldots$. It is easily seen that the distribution of the vector $\bar{b}_n$ does not converge to a stationary distribution as $n \to \infty$.

Sufficiency. Let $\bar{a}_0$ be an arbitrary nonnegative random vector. For each number $\delta > 0$ we find an $x > 0$ such that $P\{\bar{a}_0 \leqslant \bar{w}^0 + \bar{i}x\} \geqslant 1 - \delta$. Let $\bar{b}_0 = \bar{w}^0 + \bar{i}x$ and $\bar{y}$ be an arbitrary nonnegative vector. Then

$$P\{\bar{a}_n < \bar{y}\} \geqslant P\{\bar{a}_n < \bar{y};\ \bar{a}_0 \leqslant \bar{b}_0\} \geqslant P\{\bar{b}_n < \bar{y};\ \bar{a}_0 \leqslant \bar{b}_0\} \geqslant P\{\bar{b}_n < \bar{y}\} - \delta.$$

On the other hand, since $\bar{a}_0 \geqslant \bar{w}_0 = 0$ a.s. (and, consequently, $\bar{a}_n \geqslant \bar{w}_n$ a.s.), it follows that $P\{\bar{a}_n < \bar{y}\} \leqslant P\{\bar{w}_n < \bar{y}\}$ for $n = 1, 2, \ldots$.

Therefore, by virtue of Lemma 5, $\underline{\lim}\, P\{\bar{a}_n < \bar{y}\} \geqslant P\{\bar{w}^0 < \bar{y}\} - \delta$ and $\overline{\lim}\, P\{\bar{a}_n < \bar{y}\} \leqslant P\{\bar{w}^0 < \bar{y}\}$.

Since the number δ is an arbitrary positive number, the limit $\lim P\{\bar{a}_n < \bar{y}\} = P\{\bar{w}^0 < \bar{y}\}$ exists, which was desired to be proved.

Remark. After this article was sent to the editor, the author succeeded in obtaining the following strengthening of Theorem 3: The condition A is fulfilled if and only if the sequence of random vectors $\{\bar{w}_n\}$ converges strongly to a vector $\bar{w}^0$.

LITERATURE CITED

1. A. A. Borovkov, Probability Processes in Queuing Theory [in Russian], Nauka, Moscow (1972).

2. A. A. Borovkov, Asymptotic Methods in Queuing Theory [in Russian], Nauka, Moscow (1980).
3. A. A. Borovkov, "Ergodic and stability theorems for a class of stochastic equations," Teor. Veroyatn. Primen., 23, No. 2, 241-262 (1978).
4. T. Kiefer and T. Wolfowitz, "On the theory of queues with many servers," Trans. Am. Math. Soc., 78, No. 1, 1-18 (1955).
5. R. M. Loynes, "The stability of queues with nonindependent interarrival and service times," Proc. Cambr. Philos. Soc., 58, No. 3, 497-520 (1962).

A PROBLEM OF INTEGRAL GEOMETRY FOR TENSOR FIELDS AND THE ST. VENANT EQUATION

V. A. Sharafutdinov

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1. Statement of the Results

1.1. In this article we give a detailed exposition of the results announced in [1].

It is known [2, 3] that a smooth function on R^n with compact support is uniquely determined by its integrals over all lines. It is also known [4] that a differential one-form of compact support is determined by its integrals over all lines to within the differential of an arbitrary function of compact support. In this paper we generalize these results to the case of tensor fields of arbitrary degree m. More precisely, let g be a symmetric tensor field of degree m on R^n with compact support; then g is determined by the integrals over all lines of the corresponding m-form (obtained by antisymmetrizing g) to within a tensor field f of the form $f = \sigma\nabla x$, where x is an arbitrary symmetric tensor field of degree $m - 1$ with compact support, ∇ denotes the differentiation operator, and σ is the symmetrization operator.

In this context the question arises of describing the tensor fields f for which the equation

$$\sigma \nabla x = f \tag{1}$$

has a solution. We note that the analog of Eq. (1) for skew-symmetric x and f is the equation

$$\alpha \nabla x = dx = f, \tag{2}$$

where α denotes antisymmetrization. The familiar Poincare lemma asserts that for a skew-symmetric field f on R^n, Eq. (2) has a solution if and only if f is closed: $df = 0$. In this paper we derive a similar result for Eq. (1). More precisely, we define a differential operator V on the space of symmetric tensor fields f such that the condition

$$Vf = 0 \tag{3}$$

is necessary and sufficient for (1) to have a solution. We call (3) a St. Venant equation since it coincides for m = 2 with the consistency condition for deformations obtained by St. Venant [5]. In this paper we will also study the uniqueness problem for Eq. (1).

1.2. We denote by T_m ($m \geqslant 0$) the real vector space of all covariant tensors of degree m over R^n, i.e., the space of all m-linear functions $x: R^n \times \ldots \times R^n \to R$, and we write S_m for the subspace of T_m consisting of all symmetric tensors. If U is a domain in R^n then $T_m(U)$ denotes the space of all covariant tensor fields of degree m on U. In particular, $T_0 = R$ and $T_0(U)$ is the space of real-valued functions on U. We agree to write $T_{-1} = T_{-1}(U) = 0$, the zero space. If $x \in T_m(U)$, the expression $x = (x_{j_1 \ldots j_m})$ will mean that the functions $x_{j_1 \ldots j_m}(u) = x_{j_1 \ldots j_m}(u^1, \ldots, u^n)$, $u \in U$ are the components of the field x with respect to some affine (or more generally, curvilinear) coordinate system defined in U. We use a similar notation for $x \in T_m$, except that in this case the components are real numbers. We write $C^l T_m(U)$ ($l \geqslant 0$) for the subspace of $T_m(U)$ consisting of all fields whose components are l times continuously differentiable; similarly for $S_m(U)$, $C^l S_m(U)$.

We will need the following standard tensor operations: symmetrization over a group of indices, antisymmetrization with respect to two indices, and cyclization over three indices.

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