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ON THE LIMIT BEHAVIOR OF THE DISTRIBUTION OF THE CROSSING NUMBER OF A STRIP BY SAMPLE PATHS OF A RANDOM WALK

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Abstract: We find asymptotic representations for the distribution of the crossing number of an expanding strip by sample paths of a random walk in the case when the crossing number is finite with probability 1. The results are obtained under various restrictions on the rate of decrease at infinity for the distribution tails.

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Let $X_1, X_2, \dots$ be independent identically distributed random variables,

$$S_0 = 0, \quad S_n = X_1 + \dots + X_n.$$

In this article we study the crossing number of a strip $-a \le y \le b$ (where $-a \le 0 \le b$) in the coordinate plane of points (x, y) by a sample path of the random walk $\{(n, S_n), n \ge 0\}$. It is known [1] that the crossing number is finite with probability 1 if one of the series

$$\sum \frac{1}{n}\mathbf{P}(S_n > 0) < \infty \quad \text{or} \quad \sum \frac{1}{n}\mathbf{P}(S_n < 0) < \infty \tag{1}$$

converges; and for this it suffices in turn that $\mathbf{E}X_1 \ne 0$. For instance, if the first of the series (1) converges then the sample path of a random walk goes downwards, the maximum of partial sums is finite with probability 1, and the infimum of the sequence of partial sums equals $-\infty$.

Define the stopping times (possibly improper) as follows:

$$\tau_0^+ = \tau_0^- = 0, \quad \tau_i^- = \inf\{n > \tau_{i-1}^+ : S_n \le -a\}, \quad \tau_i^+ = \inf\{n > \tau_i^- : S_n \ge b\},\ i \ge 1.$$

We agree that $\inf \varnothing = \infty$.

Assume (1). Consider the random variable $\eta^{(1)}$ equal to the crossing number of the strip from below to above by a sample path of $\{(n, S_n), n \ge 0\}$. Obviously, $\mathbf{P}(\eta^{(1)} \ge k) = \mathbf{P}(\tau_k^+ < \infty)$. A similar definition applies to the random variable $\eta^{(2)}$ equal to the crossing number of the strip from above to below.

Many publications are devoted to studying the distribution of the crossing number of a strip. Some articles contain explicit formulas for the distribution of interest in various schemes of walk. In [2], the representations of the probabilities $\mathbf{P}(\eta^{(i)} \ge k)$ are given in terms of iterates of certain operators connected with factorization components of the function $1 - z\varphi(\lambda)$, where $\varphi(\lambda) = \mathbf{E}\exp\{\lambda X_1\}$. In [3], similar representations were found for a stationary process with independent increments. The explicit formulas for the distributions of random variables $\eta^{(i)}$ were studied in [4] provided that X_i are integer-valued, $\mathbf{E}X_1 \ne 0$, and the probabilities $\mathbf{P}(X_1 = k)$ and $\mathbf{P}(X_1 = -k)$ decrease in geometric progression. In [5], analogous results were obtained in the case of a geometric distribution only on one of the semiaxes. It was also observed in [4] that the probabilities $\mathbf{P}(\eta^{(i)} \ge k)$ can be always bounded from above by the corresponding probabilities for the geometric distribution. More exactly, if $\mathbf{E}X_1 < 0$ then

$$\mathbf{P}(\eta^{(1)} \ge k) \le [\mathbf{P}(S \ge a + b)]^k,$$

where $S = \sup_{n \ge 0} S_n$. In [6], the distribution of the crossing number was found for the Wiener process with drift.

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The explicit expressions for the sought distribution are available only for walks of particular type. That is why many papers in this direction are devoted to constructing various approximations, including the study of the crossing number of a strip on a finite expanding time interval. Some limit results for the crossing number of a level are presented in [7, 8]. The article [9] contains the assertion on the limit distribution of the number of zeros in the sequence S_n in the case when $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$. In [2], alongside the above-mentioned operator representations for the probabilities $\mathbf{P}(\eta^{(i)} \geq k)$, their asymptotic behavior was established provided that the strip expands without limit. This was done under the additional Cramér condition on the distribution of jumps of the random walk. The complete asymptotic expansions for the distribution of the crossing number of a strip on the finite unboundedly-increasing time interval were found in [10] for the case when the Cramér conditions hold, $\mathbf{E}X_1 = 0$, and the strip width and the length of the time interval increase in a coordinated way. Similar results were also obtained for Markov-modulated random walks [11].

The present article contains new asymptotic representations for the distribution of the crossing number of the strip from below to above provided that the strip width increases unboundedly. We separately consider the cases of slowly and rapidly decreasing distributions of X_1 at infinity. We also suppose that $\mathbf{E}X_1 < 0$. Clearly, the random variables $\eta^{(1)}$ and $\eta^{(2)}$ differ from each other at most 1. For brevity, we put $\eta = \eta^{(1)}$ in what follows.

Denote by $\mathscr{S}$ the class of subexponential distributions. We recall its definition (see, e.g., [1]). Let nonnegative independent random variables Y_1 and Y_2 have the same distribution function F. We say that $F \in \mathscr{S}_+$ if

$$\mathbf{P}(Y_1 + Y_2 \geq t) \sim 2\mathbf{P}(Y_1 \geq t) \quad \text{as } t \to \infty.$$

In particular, the class $\mathscr{S}_+$ contains the Weibull distribution $1 - F(t) = \exp\{-t^\alpha\}$, $0 < \alpha < 1$, and the distributions regular varying at infinity, i.e. $1 - F(t) = t^{-\alpha}L(t)$, where $\alpha > 0$ and the function $L(t)$ is slowly varying at infinity.

By definition, the distribution function of an arbitrary random variable Y belongs to $\mathscr{S}$ if the distribution function of the random variable $\max\{0, Y\}$ belongs to $\mathscr{S}_+$. Put

$$G(x) = \min\biggl(1, \int\limits_x^\infty \mathbf{P}(X_1 \geq t)\,dt\biggr).$$

Given a random walk satisfying the condition $\mathbf{E}X_1 < 0$, put

$$S = \sup_{n\geq 0} S_n, \quad \nu_t^- = \inf\{n \geq 1 : S_n \leq t\}, \quad \chi_t^- = S_{\nu_t^-} - t, \quad t \leq 0.$$

Let

$$G_0(v) = \frac{1}{|\mathbf{E}\chi_0^-|} \int\limits_{-\infty}^{v} \mathbf{P}(\chi_0^- < t)\,dt.$$

Theorem 1. *Suppose that $\mathbf{E}X_1 < 0$, the distribution of X_1 is nonarithmetic, and $1 - G \in \mathscr{S}$. Then for every integer $k \geq 2$ and arbitrary $a \geq 0$*

$$\mathbf{P}(\eta \geq k) = \mathbf{P}(\eta \geq 1)\biggl(\frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \geq t)(1 - G_0(a + b - t))\,dt\biggr)^{k-1} (1 + o(1)) \quad \text{as } b \to \infty.$$

PROOF. It is easily seen that, for $k \geq 2$,

$$\mathbf{P}(\eta \geq k) = \int\limits_b^\infty \int\limits_{-\infty}^{+0} \mathbf{P}(S \geq a + b - x)\mathbf{P}(\chi_{-a-y}^- \in dx)\mathbf{P}(S_{\tau_{k-1}^+} \in dy). \tag{2}$$

The integrand means that we fix the position of the walk at the point y at the time τ^+_{k-1}, i.e., at the moment of the $(k-1)$th intersection of the strip from below to above; then we fix the position x at the first (after τ^+_{k-1}) passage time of the low boundary, and then we should provide exceeding the upper boundary of the strip by the supremum of the remaining part of the trajectory. We further use the following well-known result (see, e.g., [1, Chapter 12, § 7]): if $1-G \in \mathscr{S}$ then

$$\mathbf{P}(S \ge a+b-x) = \frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b-x}^{\infty} \mathbf{P}(X_1 \ge t)\,dt(1+\psi(a+b-x)), \tag{3}$$

where $\psi(t) = o(1)$ as $t \to \infty$. Then

$$\mathbf{P}(\eta \ge k) = \frac{1}{|\mathbf{E}X_1|} \int\limits_{b}^{\infty} \int\limits_{-\infty}^{+0} \int\limits_{a+b-x}^{\infty} \mathbf{P}(X_1 \ge t)\,dt(1+\psi(a+b-x))\mathbf{P}(\chi^-_{-a-y} \in dx)\mathbf{P}(S_{\tau^+_{k-1}} \in dy).$$

Introduce $\psi_1(t) = \sup_{v \ge t} \psi(v)$. We have

$$\psi(a+b-x) \le \psi_1(a+b-x) \le \psi_1(a+b),$$

since $x \le 0$ and $\psi_1(t)$ is a nonincreasing function. Moreover, $\psi_1(t) \to 0$ as $t \to \infty$. On letting $b \to \infty$, we arrive at $\psi_1(a+b) = o(1)$. Thus,

$$\mathbf{P}(\eta \ge k) = \frac{1}{|\mathbf{E}X_1|} \int\limits_{b}^{\infty} \int\limits_{-\infty}^{+0} \int\limits_{a+b-x}^{\infty} \mathbf{P}(X_1 \ge t)\,dt\mathbf{P}(\chi^-_{-a-y} \in dx)\mathbf{P}(S_{\tau^+_{k-1}} \in dy)(1+o(1))$$

as $b \to \infty$. Changing the order of integration, we obtain

$$\mathbf{P}(\eta \ge k) = \frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \ge t) \int\limits_{b}^{\infty} \int\limits_{a+b-t}^{+0} \mathbf{P}(\chi^-_{-a-y} \in dx)\mathbf{P}(S_{\tau^+_{k-1}} \in dy)\,dt(1+o(1))$$

$$= \frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \ge t) \int\limits_{b}^{\infty} (1-\mathbf{P}(\chi^-_{-a-y} < a+b-t))\mathbf{P}(S_{\tau^+_{k-1}} \in dy)\,dt(1+o(1)).$$

The limit distribution of the excess over the infinitely remote barrier is also well known (see [1, Chapter 10, § 4] for instance). If the distribution of jumps is nonarithmetic then

$$\mathbf{P}(\chi^-_{-a-y} < v) = G_0(v) + \theta_{y+a}(v), \tag{4}$$

where $\theta_t(v) \to 0$ as $t \to \infty$ for each v. Observe that here the convergence is uniform in v, since the limit function is continuous. The above assumption $b \to \infty$ is sufficient for $y+a \to \infty$, since $y \ge b$. Therefore, for every $\varepsilon > 0$, from some $b \ge b_0$ on,

$$\left| \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \ge t) \int\limits_{b}^{\infty} \theta_{y+a}(a+b-t)\mathbf{P}(S_{\tau^+_{k-1}} \in dy)\,dt \right| \le \varepsilon \left| \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \ge t) \int\limits_{b}^{\infty} \mathbf{P}(S_{\tau^+_{k-1}} \in dy)\,dt \right|,$$

i.e., we have

$$\int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \ge t) \int\limits_{b}^{\infty} \theta_{y+a}(a+b-t)\mathbf{P}(S_{\tau^+_{k-1}} \in dy)\,dt$$

$$= o(1) \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \ge t) \int\limits_{b}^{\infty} \mathbf{P}(S_{\tau^+_{k-1}} \in dy)\,dt \quad \text{as } b \to \infty.$$

Thus,

$$\mathbf{P}(\eta \geq k) = \frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \geq t) \int\limits_{b}^{\infty} (1 - G_0(a+b-t))\mathbf{P}(S_{\tau_{k-1}^+} \in dy)\, dt (1 + o(1))$$

$$= \frac{\mathbf{P}(\tau_{k-1}^+ < \infty)}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \geq t)(1 - G_0(a+b-t))\, dt (1 + o(1)).$$

Recall that $\mathbf{P}(\tau_{k-1}^+ < \infty) = \mathbf{P}(\eta \geq k-1)$. The claim of the theorem follows from this recurrent relation.

We consider $\mathbf{P}(\eta \geq 1)$. It is clear that

$$\mathbf{P}(\eta \geq 1) = \int\limits_{-\infty}^{+0} \mathbf{P}(S \geq a+b-x)\, \mathbf{P}(\chi_{-a}^- \in dx). \tag{5}$$

Letting b tend to infinity and again using (3), we establish that

$$\mathbf{P}(\eta \geq 1) = \frac{1}{|\mathbf{E}X_1|} \int\limits_{-\infty}^{+0} \int\limits_{a+b-x}^{\infty} \mathbf{P}(X_1 \geq t)\, dt\, \mathbf{P}(\chi_{-a}^- \in dx)(1 + o(1))$$

$$= \frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \geq t) \int\limits_{a+b-t}^{+0} \mathbf{P}(\chi_{-a}^- \in dx)\, dt (1 + o(1))$$

$$= \frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \geq t)(1 - \mathbf{P}(\chi_{-a}^- < a+b-t))\, dt (1 + o(1)).$$

Assuming additionally that $a \to \infty$ and employing the asymptotics (4), we obtain

$$\mathbf{P}(\eta \geq 1) = \frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \geq t)((1 - G_0(a+b-t)))\, dt (1 + o(1)).$$

Thus, we arrive at the following assertion:

Corollary 1. *Under the conditions of Theorem* 1, *the following relation holds for every integer* $k \geq 1$*:*

$$\mathbf{P}(\eta \geq k) = \left(\frac{1}{|\mathbf{E}X_1|} \int\limits_{a+b}^{\infty} \mathbf{P}(X_1 \geq t)(1 - G_0(a+b-t))\, dt \right)^k (1 + o(1))$$

as $a \to \infty$ *and* $b \to \infty$.

We consider the situation in which the random walk jumps satisfy the unilateral Cramér condition. Introduce the following conditions:

(C_1) There is $\lambda > 0$ such that $\mathbf{E}e^{\lambda X_1} < \infty$.

(C_2) There is $q > 0$ such that

$$\varphi(q) = \mathbf{E}e^{qX_1} = 1, \quad \mathbf{E}X_1 e^{qX_1} = \varphi'(q) < \infty.$$

Put $\nu^+ = \inf\{n \geq 1 : S_n > 0\}$. Define the random variable $\chi^+ = S_{\nu^+}$ on the set $\{\nu^+ < \infty\}$. Let

$$c = \frac{\mathbf{P}(S = 0)}{q\mathbf{E}(\chi^+ \exp\{q\chi^+\}; \nu^+ < \infty)}, \qquad d = \frac{1}{|\mathbf{E}\chi_0^-|} \int\limits_{-\infty}^{0} e^{qx}\mathbf{P}(\chi_0^- < x)\, dx.$$

Here $c < 1$ (see [1, Chapter 12, § 7]) and, obviously, $d < 1$.

Theorem 2. *Let $\mathbf{E}X_1 < 0$, let the distribution of X_1 be nonarithmetic, and let the conditions (C_1) and (C_2) be satisfied. Then the following relation takes place for every integer $k \geq 2$ and arbitrary $a \geq 0$ as $b \to \infty$:*

$$\mathbf{P}(\eta \geq k) = \mathbf{P}(\eta \geq 1)(cd)^{k-1} e^{-q(k-1)(a+b)}(1+o(1)).$$

PROOF. We again use (2), but, under the conditions of the theorem, the asymptotics of the distribution of the supremum is different; namely,

$$\mathbf{P}(S \geq x) = ce^{-qx}(1 + r(x)), \tag{6}$$

where $r(x) \to 0$ as $x \to \infty$ (see [1, Chapter 12, § 7]). Various expressions for the constant c are given in the same book. As in the proof of Theorem 1, from (2) we deduce that

$$\begin{aligned}\mathbf{P}(\eta \geq k) &= \int_b^{\infty} \int_{-\infty}^{+0} ce^{-q(a+b-x)}(1 + r(a+b-x))\mathbf{P}(\chi^-_{-a-y} \in dx)\mathbf{P}(S_{\tau^+_{k-1}} \in dy) \\ &= ce^{-q(a+b)} \int_b^{\infty} \int_{-\infty}^{+0} e^{qx}\mathbf{P}(\chi^-_{-a-y} \in dx)\mathbf{P}(S_{\tau^+_{k-1}} \in dy)(1+o(1))\end{aligned} \tag{7}$$

as $b \to \infty$. In view of weak convergence (4), we have

$$\int_{-\infty}^{+0} e^{qx}\mathbf{P}(\chi^-_{-a-y} \in dx) \to d = \frac{1}{|\mathbf{E}\chi_0^-|} \int_{-\infty}^{0} e^{qx}\mathbf{P}(\chi_0^- < x)\, dx \quad \text{as} \quad b \to \infty.$$

Therefore, (4) transforms into

$$\mathbf{P}(\eta \geq k) = cde^{-q(a+b)} \int_b^{\infty} \mathbf{P}(S_{\tau^+_{k-1}} \in dy)(1+o(1)) = \mathbf{P}(\eta \geq k-1)cde^{-q(a+b)}(1+o(1)).$$

The theorem is proven.

Return to (5). Under our conditions, we have

$$\mathbf{P}(\eta \geq 1) = \int_{-\infty}^{+0} \mathbf{P}(S \geq a+b-x)\, \mathbf{P}(\chi^-_{-a} \in dx) = ce^{-q(a+b)} \int_{-\infty}^{+0} e^{qx}\, \mathbf{P}(\chi^-_{-a} \in dx)(1+o(1)). \tag{8}$$

If, additionally, $a \to \infty$, then

$$\int_{-\infty}^{+0} e^{qx}\, \mathbf{P}(\chi^-_{-a} \in dx) \to d.$$

Thus, we have proven

Corollary 2. *Let the conditions of Theorem 2 be satisfied. Assuming that $a \to \infty$ and $b \to \infty$, we have*

$$\mathbf{P}(\eta \geq k) = (cd)^k e^{-qk(a+b)}(1+o(1)) \tag{9}$$

for every integer $k \geq 1$.

The asymptotic representation (9) was established earlier in [2] under stronger requirements on the distribution of X_1.

Remark 1. Calculations in Theorems 1 and 2 can be simplified, if

$$\mathbf{P}(X_1 < t) = \alpha \exp\{\beta t\}, \quad t \le 0.$$

In this case $\mathbf{P}(\chi_0^- < t) = G_0(t) = \exp\{\beta t\}$, $t \le 0$, and $d = \beta/(\beta + q)$.

Remark 2. If a does not increase then, for calculating the right-hand side of (8), we may use the following relation (see [2]) on assuming $|z| < 1$ and $\operatorname{Re}\lambda \ge 0$:

$$\mathbf{E}(z^{\nu_{-a}^-} \exp\{\lambda \chi_{-a}^-\}) = R_-(z,\lambda)[R_-^{-1}(z,\lambda)]^{(-\infty,-a]}. \tag{10}$$

Here $R_-(z,\lambda)$ is a negative component of the factorization

$$1 - z\varphi(\lambda) = R_-(z,\lambda)R_+(z,\lambda)$$

(for details see, for instance, [1, Chapter 12]). In (10) we have used the notation

$$\left[\int_{-\infty}^{\infty} e^{\lambda x} dH(x)\right]^A = \int_A e^{\lambda x} dH(x), \quad \text{where} \int_{-\infty}^{\infty} |dH(x)| < \infty, \quad \operatorname{Re}\lambda = 0.$$

The left-hand side of (10) exists for $z = 1$ and is continuous at this point; therefore,

$$\int_{-\infty}^{+0} e^{qx}\mathbf{P}(\chi_{-a}^- \in dx) = \lim_{z\to 1} \mathbf{E}(z^{\nu_{-a}^-} \exp\{q\chi_{-a}^-\}) = \lim_{z\to 1} R_-(z,\lambda)\big[R_-^{-1}(z,\lambda)\big]^{(-\infty,-a]}|_{\lambda=q}.$$

Finding of the factorization component $R_-(z,\lambda)$ in the explicit form is available for a wide class of random walks; for details see [1, Chapter 12].

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