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ON THE SOJOURN TIME OF A RANDOM WALK IN A STRIP

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Abstract: We obtain asymptotic representations for the triple transforms of the joint distribution of the sojourn time of a random walk in a strip (as well as in a half-plane) in n steps and of the location at time n under the condition of unboundedly moving-off boundaries of the sets. The Cramér type conditions are imposed on the distribution of jumps.

Keywords: random walk, sojourn time in a strip, factorization identities, moment generating functions, asymptotic analysis

1. Statement of the Problem. The Main Result

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables, $S_n = X_1 + \dots + X_n$, $n \geq 1$. Given a Borel set $D \subset R$, we introduce

$$T_n = T_n(D) = \sum_{k=1}^{n} I_{\{S_k \in D\}},$$

where $I_A(\omega) = 1$ if $\omega \in A$, and $I_A(\omega) = 0$ otherwise. Thus, T_n is the sojourn time of a random walk in D on the time interval $[1, \dots, n]$, i.e. the number of points k, $1 \leq k \leq n$, such that $S_k \in D$.

Throughout the sequel we assume that $D = [-a, b]$, where $-a \leq b$. Moreover, we speak about the *two-sided boundary problem* if both numbers a and b are finite. If $a = \infty$ and $b < \infty$ or $a < \infty$ and $b = \infty$ then we deal with the *one-sided boundary problem*.

The final goal of our research is to find the probabilities $\mathbf{P}(T_n = k)$ and study their behavior for large n, including the case in which the set D itself depends on n. There are various approaches to this problem. The combinatorial methods of [1, 2] can be applied to simple random walks of a special form. Most of publications are devoted to the case $D = (0, \infty)$ and the arcsine law connected with this case (for instance, see [3, 4]). The available limit theorems for the sojourn time are based on the convergence of the distributions of functionals of the trajectories of a random walk to the distribution of the corresponding functionals of limit processes. A rather comprehensive bibliography and the results in this direction of research are given in [5, 6].

Certainly, the study of the sojourn time (by its type) relates to the boundary problems for random walks, i.e. to finding the probabilities connected with the mutual disposition of the trajectories of a random walk and the boundary of some set. It is well known that, for many boundary problems, the in-depth results, including the complete asymptotic expansions of probabilities, can be obtained on using factorization identities (see [7–9] and the bibliography therein).

The factorization method of analysis is well developed by now; which explains our natural intention to apply the technique to the study of the sojourn time so obtaining some new results. Some steps in this direction were made in [10–12].

This method usually consists of several stages. At the first step, some factorization identities are found for the double or triple transforms of the sought boundary functional distribution. As a rule, they do not lead to expressions applicable for further inversion in the general case. For this reason, some scheme of asymptotic analysis is performed at the second stage (e.g., if the boundaries of the set

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under consideration move away). At this step, the principal part of the already-found transform can be isolated simultaneously with the estimation of the error of this approximation; the error turns out to be negligible under wide conditions. This stage is somewhat complicated since it involves rather delicate properties of factorization components. The third stage contains the inversion (as a rule, asymptotic) of the principal parts of the transforms. This can be fulfilled by the contour integration procedure with using the saddle-point method [7, 9] or by direct calculations [8].

We note that, for random walks under the Cramér condition on jumps, the detailed study of the analytic properties of the factorization components was carried out in [7]. Much research on the boundary crossing problems is based on the results of [7].

The above-mentioned papers [10–12] correspond to the first stage of the study: finding factorization representations for the multiple moment generating functions of the sojourn time. They use rather complicated technique of the matrix factorization which looks inefficient for the further inversion of the resultant representations. Some remarks in this connection are in [13]. Thus, the study of the sojourn time in terms of the well-known one-dimensional factorization seems actual as before.

The first stage of this work was done in [13]; the identities for the triple transforms of the joint distribution of the pair (T_n, S_n) were found there in terms of the ordinary factorization in one-sided and two-sided boundary problems. These identities provide explicit expressions for the triple transforms in one-sided boundary problems but the sought transforms enter the identities implicitly in the case of two boundaries. In both cases, the results were stated in terms of special projection operators of factorization components. These operators can easily be calculated if the distribution of X_1 has exponential density on a half-line in one-sided boundary problems or on both half-lines in two-sided problems. For random walks with general distribution of jumps, the identities do not give any compact expressions suitable for further inversion. We therefore apply asymptotic analysis to the factorization representations obtained in [13], provided that $a \to \infty$ and $b \to \infty$, in order to find the main terms with a rather simple structure.

Thus, the present article is a continuation of [13] and corresponds to the second stage of the study. The main results of this article are the asymptotic representations for the triple transforms of the joint distribution of the pair (T_n, S_n) in one-sided and two-sided boundary problems and, as a corollary, for the moment generating functions for the distribution of T_n (Theorem 1). We isolate relatively simple main terms and estimate the remainders that turn out exponentially small as compared with the principal parts. In addition, the Cramér type conditions are imposed on the distribution of X_1.

The asymptotic behavior of the coefficients of a power series is usually determined by the behavior of the sum of this series in a neighborhood of the unity. The asymptotic analysis of the distribution of the first exit time from a half-plane or a strip (see [7, 8]) shows that, even in the case of complete asymptotic expansions of the probabilities under the Cramér condition, it suffices to determine the behavior of the corresponding moment generating function near the unity. The modulus of the moment generating function is exponentially small on the remaining part of the unit circle. Therefore, we below find the asymptotic representation of the function

$$f_{a,b}(z,u,\lambda) = \sum_{n=1}^{\infty} z^n \mathbf{E}(u^{T_n([-a,b])} e^{\lambda S_n})$$
$$= \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \int_{-\infty}^{\infty} e^{\lambda x} \mathbf{P}(T_n([-a,b]) = k, S_n \in dx)$$

only near the unity in z and u.

To avoid long statements, in this section we give an asymptotic representation for $f_{a,b}(z,u,0)$. To this end, we need some notation. They all are connected with factorization components whose definition should be recalled as well. First, introduce the ladder epochs $\eta_\pm$ and the ladder heights $\chi_\pm$ as follows:

$$\eta_- = \inf\{n \geq 1 : S_n < 0\}, \quad \eta_+ = \inf\{n \geq 1 : S_n > 0\}, \quad \chi_\pm = S_{\eta_\pm}.$$

Here we put $\eta_+ = \infty$ if $S_n \leq 0$ for all n and $\eta_- = \infty$ if $S_n \geq 0$ for all n. We do not define the quantities $\chi_\pm$ on the events $\{\eta_\pm = \infty\}$.

Let $R_\pm(z,\lambda) = 1 - \mathbf{E}(z^{\eta_\pm}\exp\{\lambda\chi_\pm\};\eta_\pm < \infty)$ for $|z| \le 1$ and $\operatorname{Re}\lambda = 0$. These functions are components of the well-known factorization (for instance, see [14])

$$1 - z\varphi(\lambda) = R_+(z,\lambda)R_-(z,\lambda)R_0(z), \tag{1}$$

where

$$R_0(z) = \exp\Big\{ -\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbf{P}(S_n = 0)\Big\}.$$

It is known also that the following representations are valid for $|z| < 1$ and $\operatorname{Re}\lambda = 0$:

$$R_-(z,\lambda) = \exp\Big\{ -\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbf{E}(\exp\{\lambda S_n\}; S_n < 0)\Big\},$$

$$R_+(z,\lambda) = \exp\Big\{ -\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbf{E}(\exp\{\lambda S_n\}; S_n > 0)\Big\}.$$

Throughout the sequel, we suppose that the following Cramér type conditions are satisfied:

A1. The distribution of X_1 contains an absolutely continuous component.

A2. $|\varphi(\lambda)| < \infty$ for $-\gamma \le \operatorname{Re}\lambda \le \beta$, where $\gamma \ge 0$, $\beta \ge 0$, $\beta + \gamma > 0$, and $\mathbf{E}|X_1| < \infty$.

Below the values of γ and β will be specified each time.

If $\mu_1 = \mathbf{E}X_1 = 0$ and $\gamma > 0$, $\beta > 0$ then, as follows from condition A2, for some $\delta_1 > 0$ the function $1 - z\varphi(\lambda)$ has exactly the two real zeros $\lambda_\pm(z)$, $z \in [1-\delta_1, 1]$, $\lambda_-(z) \le 0 \le \lambda_+(z)$. The functions $\lambda_\pm(z)$ admit analytical continuation in some δ-neighborhood of the interval $[1-\delta_1, 1]$ with a cut along the ray $z \ge 1$. Moreover, $\lambda_\pm(z)$ retain the zeros of the function $1 - z\varphi(\lambda)$. If $\mu_1 \ge 0$ and $\beta > 0$ then, for the same z, we can guarantee only existence of the zero $\lambda_+(z) \ge 0$. This zero also exists under the condition $\mu_1 < 0$, $\beta > 0$, and $\varphi(\beta) > 1$. Analogously, the zero $\lambda_-(z)$ always exists if $\mu_1 \le 0$ and $\gamma > 0$ as well as for $\mu_1 > 0$, $\gamma > 0$, and $\varphi(\gamma) > 1$. If condition A2 is satisfied then (1) is valid in the domain $-\gamma \le \operatorname{Re}\lambda \le \beta$. The functions $\lambda_\pm(z)$, provided that they exist, are zeros of the factorization components too: $R_\pm(z, \lambda_\pm(z)) = 0$. These and the other facts to be used in the sequel are presented in detail in [7]. Put $\lambda_\pm = \lambda_\pm(1)$. Then, clearly, $\lambda_\pm = 0$ for $\mu_1 = 0$; $\lambda_- = 0$, $\lambda_+ > 0$ for $\mu_1 < 0$; and $\lambda_- < 0$, $\lambda_+ = 0$ for $\mu_1 > 0$.

All subsequent assertions that use the functions $\lambda_\pm(s)$ and $\lambda_\pm(z)$ extend only to those situations when these functions exist in accord with condition A2 and the above remarks.

Put

$$\begin{gathered} V(z,s,\lambda) = \frac{R_+(s,\lambda)R_+(z,\lambda_+(s))}{(\lambda - \lambda_+(s))R'_+(s,\lambda_+(s))R_+(z,\lambda)}, \\ U(z,s,\lambda) = \frac{R_-(s,\lambda)R_-(z,\lambda_-(s))}{(\lambda - \lambda_-(s))R'_-(s,\lambda_-(s))R_-(z,\lambda)}, \\ H_1(z,s) = U(z,s,\lambda_+(s)), \quad H_2(z,s) = V(z,s,\lambda_-(s)), \\ H(z,s) = H_1(z,s)H_2(z,s), \quad \mu(s) = e^{\lambda_-(s)-\lambda_+(s)}, \end{gathered} \tag{2}$$

and let $L_\delta = \{s : |s| < 1, |s-1| < \delta\}$.

Theorem 1. *Let $\mathbf{E}X_1 = 0$, $a \ge 0$, and $b \ge 0$ and let conditions* A1 *and* A2 *be satisfied with $\gamma > 0$ and $\beta > 0$. Then there are $\delta > 0$ and $\varepsilon > 0$ such that the following representation takes place for $z \in L_\delta$ and $s = zu \in L_\delta$:*

$$\begin{aligned} (1-s)(1 + f_{a,b}(z,u,0)) = 1 - U(z,s,0)&\frac{1 - H_2(z,s)\mu^b(s)}{1 - H(z,s)\mu^{b+a}(s)}e^{\lambda_-(s)a} \\ -V(z,s,0)\frac{1 - H_1(z,s)\mu^a(s)}{1 - H(z,s)\mu^{b+a}(s)}&e^{-\lambda_+(s)b} + (u-1)\Delta(z,s), \end{aligned} \tag{3}$$

where $|\Delta(z,s)| \le C_1e^{-\varepsilon a} + C_2e^{-\varepsilon b}$ for $|z| < |s|$.

Here and in the sequel, the symbols $C, C_1, C_2, \dots$ denote various constants; moreover, the same symbol may correspond to different constants in different expressions.

Remark. 1. We observe that the values of $a = \infty$ or $b = \infty$ in (3) are not excluded. Considering $|\mu(s)| < 1$, $\operatorname{Re}\lambda_-(s) < 0$, and $\operatorname{Re}\lambda_+(s) > 0$ for $s \in L_\delta$ with small δ, from (3) we obtain the corresponding representations for one-sided boundary problems. For instance,

$$(1-s)(1+f_{\infty,b}(z,u,0)) = 1 - V(z,s,0)e^{-\lambda_+(s)b} + (u-1)\Delta(z,s),$$

where $|\Delta(z,s)| \le Ce^{-\varepsilon b}$ for $|z| < |s|$.

2. As will be shown below, the functions in (3)

$$\left|\frac{1-H_2(z,s)\mu^b(s)}{1-H(z,s)\mu^{b+a}(s)}\right|, \quad \left|\frac{1-H_1(z,s)\mu^a(s)}{1-H(z,s)\mu^{b+a}(s)}\right|$$

are separated from 0 uniformly in a and b; and the convergence $\lambda_\pm(s) \to 0$ takes place as $s \to 1$. For this reason, the remainder $\Delta(z,s)$ in (3) decreases exponentially rapidly with the growth of a and b in comparison with the principal terms.

3. The representation (3) is essentially simplified if we assume that the distribution of X_1 has density of the form

$$f(t) = \begin{cases} c_1 e^{-\alpha_1 t}, & t > 0, \\ c_2 e^{\alpha_2 t}, & t \le 0, \end{cases} \tag{4}$$

where $\alpha_i > 0$ and $c_1\alpha_2 + c_2\alpha_1 = \alpha_1\alpha_2$. In this case $\Delta(z,s) \equiv 0$,

$$\varphi(\lambda) = \frac{\lambda(c_2 - c_1) - \alpha_1\alpha_2}{(\lambda - \alpha_1)(\lambda + \alpha_2)}$$

(the function exists in the strip $-\alpha_2 < \operatorname{Re}\lambda < \alpha_1$), and

$$1 - z\varphi(\lambda) = \frac{\lambda^2 - \lambda(\alpha_1 - \alpha_2 + z(c_2 - c_1)) + \alpha_1\alpha_2(z-1)}{(\lambda - \alpha_1)(\lambda + \alpha_2)} = \frac{(\lambda - \lambda_+(z))(\lambda - \lambda_-(z))}{(\lambda - \alpha_1)(\lambda + \alpha_2)}.$$

In view of the uniqueness properties for (1), we can put

$$R_+(z,\lambda) = \frac{\lambda - \lambda_+(z)}{\lambda - \alpha_1}, \quad R_-(z,\lambda) = \frac{\lambda - \lambda_-(z)}{\lambda + \alpha_2}, \tag{5}$$

and, using rather routine calculations, obtain the formulas

$$H_1(z,s) = \frac{\lambda_-(s) - \lambda_-(z)}{\lambda_+(s) - \lambda_-(z)}, \quad H_2(z,s) = \frac{\lambda_+(s) - \lambda_+(z)}{\lambda_-(s) - \lambda_+(z)}, \tag{6}$$

$$V(z,s,0) = \frac{\lambda_+(z) - \lambda_+(s)}{\lambda_+(z)}, \quad U(z,s,0) = \frac{\lambda_-(z) - \lambda_-(s)}{\lambda_-(z)}. \tag{7}$$

2. Preliminary Results

The results of [13] are a starting point of our studies; therefore, we give their statements.

Let $D = [-a,b]$. Introduce the functions

$$Q_0(z,u,\lambda) = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \int\limits_{[-a,b]} e^{\lambda x}\mathbf{P}(T_n = k, S_n \in dx),$$

$$Q_1(z,u,\lambda) = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \int\limits_{(-\infty,-a)} e^{\lambda x}\mathbf{P}(T_n = k, S_n \in dx),$$

$$Q_2(z,u,\lambda) = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \int\limits_{(b,\infty)} e^{\lambda x}\mathbf{P}(T_n = k, S_n \in dx),$$

where $|z| < 1$, $|uz| < 1$, and $\operatorname{Re}\lambda = 0$. It is established in [13] that

$$u(Q_1(z,u,\lambda) + Q_2(z,u,\lambda))(1 - z\varphi(\lambda)) + (1 + Q_0(z,u,\lambda))(1 - zu\varphi(\lambda)) = 1. \tag{8}$$

Rewrite (8) as follows:

$$\begin{aligned} &1 + Q_1(z,u,\lambda) + Q_2(z,u,\lambda) + Q_0(z,u,\lambda) \\ &= \frac{1}{1 - zu\varphi(\lambda)}(1 + (1-u)(Q_1(z,u,\lambda) + Q_2(z,u,\lambda))). \end{aligned}$$

Clearly, the left-hand side of this equality is equal to $1 + f_{a,b}(z,u,\lambda)$, i.e.

$$1 + f_{a,b}(z,u,\lambda) = \frac{1}{1 - zu\varphi(\lambda)}(1 + (1-u)(Q_1(z,u,\lambda) + Q_2(z,u,\lambda))). \tag{9}$$

Thus, to find $f_{a,b}(z,u,\lambda)$, it suffices to know Q_i, $i = 1,2$. In [13] the equation (8) was solved by a Wiener–Hopf type method; in result, the representations of Q_i, $i = 1,2$, were found via the factorization components of $1 - z\varphi(\lambda)$ and $1 - zu\varphi(\lambda)$. To formulate this result, we introduce some additional notation.

For every function g of the form

$$g(z,s,\lambda) = \int_{-\infty}^{\infty} e^{\lambda x} dG_{z,s}(x), \quad \text{where} \int_{-\infty}^{\infty} |dG_{z,s}(x)| < \infty,$$

we define the operators $L_\pm$ as follows:

$$(L_+g)(z,s,\lambda) = \frac{R_+(s,\lambda)}{R_+(z,\lambda)}\left[\frac{R_+(z,\lambda)}{R_+(s,\lambda)} g(z,s,\lambda)\right]^{(b,\infty)},$$

$$(L_-g)(z,s,\lambda) = \frac{R_-(s,\lambda)}{R_-(z,\lambda)}\left[\frac{R_-(z,\lambda)}{R_-(s,\lambda)} g(z,s,\lambda)\right]^{(-\infty,-a)}.$$

Here we use the notation

$$\left[\int_{-\infty}^{\infty} e^{\lambda x} dG(x)\right]^{A} = \int_{A} e^{\lambda x} dG(x).$$

The so-defined operators themselves depend on z and s; but for brevity we do not emphasize this fact in their notation.

Put $s = zu$, $q_i(z,s,\lambda) \equiv Q_i(z, s/z, \lambda)$, and $h(z,s,\lambda) \equiv z(s-z)^{-1}$.

The following theorem was proven in [13].

Theorem 2. *Let $D = [-a, b]$, $-a \le 0 \le b$. Then the following representations are valid for $|z| < 1$, $|s| < 1$, and $\operatorname{Re}\lambda = 0$:*

$$\begin{aligned} q_1(z,s,\lambda) &= (L_-h)(z,s,\lambda) - (L_-L_+h)(z,s,\lambda) + (L_-L_+q_1)(z,s,\lambda), \\ q_2(z,s,\lambda) &= (L_+h)(z,s,\lambda) - (L_+L_-h)(z,s,\lambda) + (L_+L_-q_2)(z,s,\lambda). \end{aligned} \tag{10}$$

Corollary 1. *For $D = [-a, \infty)$, $-a \le 0$, we have*

$$q_1(z,s,\lambda) = (L_-h)(z,s,\lambda) \tag{11}$$

and, similarly, for $D = (-\infty, b]$, $b \ge 0$,

$$q_2(z,s,\lambda) = (L_+h)(z,s,\lambda). \tag{12}$$

Although the formulas (10) determine iteration processes for finding q_i, they lead to sufficiently simple explicit expressions for q_i only in the particular cases when the factorization components, for instance, are linear-fractional functions (see [13] and also Theorem 4 below). However, simple expressions for q_i are unavailable if X_1 has distribution of a general form. Therefore, for the further inversion of the moment generating functions we have to use their asymptotic representations which will be found below. If $a \to \infty$, $b \to \infty$, and the Cramér condition holds (see below) then the expressions of the form $(L_\pm g)(z, s, \lambda)$ can be approximated by simpler expressions in a neighborhood of the unity (in variables z and s) with exponentially small error of this approximation. Inserting the resultant approximating expressions into (10) leads to the asymptotic representations for q_1 and q_2 and then for $f_{a,b}(z, u, \lambda)$. This all will be done in the following sections.

As is seen from the formulas (10), we pass from the variables z and u to the variables z and s. In many cases, this fact justifies the passage from the double transform over $T_n(D)$ to the transform over $T_n(\overline{D})$, since

$$\sum_{n=1}^{\infty} z^n \mathbf{E} u^{T_n(D)} = \sum_{n=1}^{\infty} z^n \mathbf{E} u^{n-T_n(\overline{D})} = \sum_{n=1}^{\infty} s^n \mathbf{E} v^{T_n(\overline{D})}, \tag{13}$$

where $s = zu$ and $v = u^{-1}$.

3. Asymptotic Analysis of the Operators $L_\pm$

Denote by Π the set of functions g of the form

$$g(\lambda) = \int_{-\infty}^{\infty} e^{\lambda y} dG(y), \quad \operatorname{Re}\lambda = 0, \quad \|g\| = \int_{-\infty}^{\infty} |dG(y)| < \infty.$$

In line with [8], given arbitrary $t \in \mathbb{R}$, introduce the sets

$$\Pi(t) = \{g(\lambda) : g(\lambda + t) \in \Pi\},$$

$$\Pi_-(t) = \Big\{ g \in \Pi(t) : g(\lambda) = \int_{(-\infty,0]} e^{\lambda y} dG(y),\ \operatorname{Re}\lambda = t \Big\},$$

$$\Pi_+(t) = \Big\{ g \in \Pi(t) : g(\lambda) = \int_{[0,\infty)} e^{\lambda y} dG(y),\ \operatorname{Re}\lambda = t \Big\}$$

and, for $g \in \Pi(t)$ and $g \in \Pi_\pm(t)$, put $\|g\|_t = \int e^{ty} |dG(y)|$ with the obvious agreements on domains of integration. We may assume that the variables z and s are independent in the formulations of the forthcoming Lemmas 1–3.

First of all we need Lemma 1 of [8] and its modification.

Lemma 1. *Let conditions* A1 *and* A2 *hold. Then there are* $\delta > 0$ *and* $\varepsilon > 0$ *such that, for*

$$s \in L_\delta = \{s : |s| < 1,\ |s - 1| < \delta\}, \quad |z| \le 1,$$

there exists at least one of the functions $\lambda_\pm(s)$*, and the representations*

$$\frac{R_\pm(z, \lambda)}{R_\pm(s, \lambda)} = \frac{R_\pm(z, \lambda_\pm(s))}{(\lambda - \lambda_\pm(s)) R'_\pm(s, \lambda_\pm(s))} + \psi_\pm(z, s, \lambda)$$

are valid in which the derivative is calculated in the variable λ, $\psi_\pm(z, s, \lambda) \in \Pi_\pm(\lambda_\pm \pm \varepsilon)$*; moreover, the norms* $\|\psi_\pm(z, s, .)\|_{\lambda_\pm \pm \varepsilon}$ *of these functions are bounded uniformly in* $s \in L_\delta$ *and* $|z| \le 1$.

Proof. Put

$$w_+(s, \lambda) = \frac{R_+(s, \lambda)(\lambda + \gamma + 1)}{\lambda - \lambda_+(s)}, \quad w_-(s, \lambda) = \frac{R_-(s, \lambda)(\lambda - \beta - 1)}{\lambda - \lambda_-(s)}.$$

It is known [8] that

$$w_+^{\pm 1}(s,\lambda) \in \Pi_+(\lambda_+ + \varepsilon), \quad w_-^{\pm 1}(s,\lambda) \in \Pi_-(\lambda_- - \varepsilon), \quad s \in L_\delta,$$

for some $\delta > 0$ and $\varepsilon > 0$, and the norms of these functions are bounded uniformly in $s \in L_\delta$. Extracting the singularity at $\lambda = \lambda_+(s)$, we obtain

$$\frac{R_+(z,\lambda)}{R_+(s,\lambda)} = \frac{R_+(z,\lambda)(\lambda+\gamma+1)}{(\lambda-\lambda_+(s))w_+(s,\lambda))}$$
$$= \frac{R_+(z,\lambda_+(s))(\lambda+\gamma+1)}{(\lambda-\lambda_+(s))w_+(s,\lambda)} + \frac{(R_+(z,\lambda)-R_+(z,\lambda_+(s)))(\lambda+\gamma+1)}{(\lambda-\lambda_+(s))w_+(s,\lambda)}.$$

Clearly,

$$\psi_+^{(1)}(z,s,\lambda) \equiv \frac{(R_+(z,\lambda)-R_+(z,\lambda_+(s)))(\lambda+\gamma+1)}{(\lambda-\lambda_+(s))w_+(s,\lambda)} \in \Pi_+(\lambda_+ + \varepsilon).$$

Moreover,

$$\frac{\lambda+\gamma+1}{(\lambda-\lambda_+(s))w_+(s,\lambda)} = w_+^{-1}(s,\lambda)$$
$$+\frac{\lambda_+(s)+\gamma+1}{\lambda-\lambda_+(s)}\left[w_+^{-1}(s,\lambda_+(s)) + (\lambda-\lambda_+(s))\frac{w_+^{-1}(s,\lambda)-w_+^{-1}(s,\lambda_+(s))}{\lambda-\lambda_+(s)}\right]$$
$$= \frac{1}{(\lambda-\lambda_+(s))R_+'(s,\lambda_+(s))} + w_+^{-1}(s,\lambda) + (\lambda_+(s)+\gamma+1)\frac{w_+^{-1}(s,\lambda)-w_+^{-1}(s,\lambda_+(s))}{\lambda-\lambda_+(s)}$$
$$\equiv \frac{1}{(\lambda-\lambda_+(s))R_+'(s,\lambda_+(s))} + \psi_+^{(2)}(s,\lambda),$$

where $\psi_+^{(2)}(s,\lambda) \in \Pi_+(\lambda_+ + \varepsilon)$ as well as $w_+^{-1}(s,\lambda)$ (see [7]). We are left with putting

$$\psi_+(z,s,\lambda) = \psi_+^{(1)}(z,s,\lambda) + R_+(z,\lambda_+(s))\psi_+^{(2)}(s,\lambda).$$

The assertion for $R_-(z,\lambda)R_-^{-1}(s,\lambda)$ can be proven similarly. The lemma is proven.

In what follows, the number $\varepsilon > 0$ is chosen according to the claim of the above lemma. Observe also that the equality $\psi_+(s,s,\lambda) \equiv 1$ follows from the definition of $\psi_+(z,s,\lambda)$. For this reason, $\psi_+(z,s,\lambda)$ can be representable as

$$\psi_+(sv,s,\lambda) = 1 + (v-1)\psi_+^{(0)}(sv,s,\lambda), \quad z = sv,$$

with the same properties of analyticity in λ for $\psi_+^{(0)}(z,s,\lambda)$ and $\psi_+(z,s,\lambda)$. The case of $\psi_-(z,s,\lambda)$ is exactly the same. We will use this fact in proving the following lemmas.

Lemma 2. *Let conditions* A1 *and* A2 *hold. In addition, we assume that* $\beta > 0$ *and* $\varphi(\beta) > 1$ *if* $\mathbf{E}X_1 < 0$. *Then, for every function* $g(\lambda) = \int_{(-\infty,0]} e^{\lambda y} dG(y) \in \Pi_-(0)$ *there exists* $\delta > 0$ *such that the following representations take place for* $|z| < 1$, $s \in L_\delta$, *and* $\operatorname{Re}\lambda \le \lambda_+$*:*

$$(L_+g)(z,s,\lambda) = V(z,s,\lambda)e^{(\lambda-\lambda_+(s))b}g(\lambda_+(s)) + \frac{(z-s)R_+(s,\lambda)}{sR_+(z,\lambda)} \int\limits_{(b,\infty)} e^{\lambda y}\, d\varphi_{z,s}(y),$$

where $V(z,s,\lambda)$ *is defined in* (2) *and, for all* $x \ge 0$, *we have*

$$\int\limits_{(b+x,\infty)} e^{\tau y}|d\varphi_{z,s}(y)| \le Ce^{-(\lambda_+ + \varepsilon - \tau)(b+x)}\|g\|_\tau, \quad \tau \le \lambda_+ + \varepsilon,$$

$$\left|\int\limits_{(b+x,\infty)} e^{\tau y} d\varphi_{z,s}(y)\right| \le Ce^{-(\lambda_+ + \varepsilon - \operatorname{Re}\tau)(b+x)}|g(\tau)|, \quad \operatorname{Re}\tau \le \lambda_+ + \varepsilon.$$

PROOF. We make use of Lemma 1 and the remark after its proof. If $s \in L_\delta$ then we have the inequality $\operatorname{Re}\lambda_+(s) > \lambda_+$ and so, for $\operatorname{Re}\lambda \le \lambda_+$, obtain

$$\left[\frac{g(\lambda)}{\lambda-\lambda_+(s)}\right]^{[b,\infty)} = -\int_b^\infty e^{\lambda y} \int_{(-\infty,0]} e^{-\lambda_+(s)(y-t)}\, dG(t)dy$$

$$= -g(\lambda_+(s)) \int_b^\infty e^{(\lambda-\lambda_+(s))y}\, dy = \frac{g(\lambda_+(s))e^{(\lambda-\lambda_+(s))b}}{\lambda-\lambda_+(s)}.$$

Therefore,

$$(L_+g)(z,s,\lambda) = \frac{R_+(s,\lambda)}{R_+(z,\lambda)}\left[\left(\frac{R_+(z,\lambda_+(s))}{(\lambda-\lambda_+(s))R'_+(s,\lambda_+(s))} + 1 + (z-s)s^{-1}\psi_+^{(0)}(z,s,\lambda)\right)g(\lambda)\right]^{(b,\infty)}$$

$$= V(z,s,\lambda)e^{(\lambda-\lambda_+(s))b}g(\lambda_+(s)) + \frac{(z-s)R_+(s,\lambda)}{sR_+(z,\lambda)} \int_{(b,\infty)} e^{\lambda y} d_y \int_{(-\infty,0]} \Psi_{z,s}(y-t)\, dG(t),$$

where $\Psi_{z,s}$ satisfies the relation

$$\psi_+^{(0)}(z,s,\lambda) = \int_0^\infty e^{\lambda y}\, d\Psi_{z,s}(y), \quad \operatorname{Re}\lambda \le \lambda_+ + \varepsilon.$$

Put

$$\varphi_{z,s}(y) = \int_{(-\infty,0]} \Psi_{z,s}(y-t)\, dG(t).$$

From the properties of $\psi_+(z,s,\lambda)$ in Lemma 1 it follows that

$$\left|\int_v^\infty e^{\tau y} d\Psi_{z,s}(y)\right| \le \int_v^\infty e^{\tau y}|d\Psi_{z,s}(y)| = \int_v^\infty e^{(\lambda_++\varepsilon)y-(\lambda_++\varepsilon-\tau)y}|d\Psi_{z,s}(y)| \le Ce^{-(\lambda_++\varepsilon-\tau)v}$$

for $\tau \le \lambda_+ + \varepsilon$ uniformly in s and z (see Lemma 2 in [7]). Therefore,

$$\int_{(b+x,\infty)} e^{\tau y}|d\varphi_{z,s}(y)| \le \int_{(b+x,\infty)} e^{\tau y} d_y \int_{(-\infty,0]} |\Psi_{z,s}(y-t)|\,|dG(t)|$$

$$= \int_{(b+x,\infty)} e^{\tau u}|d\Psi_{z,s}(u)| \int_{(-\infty,0]} e^{\tau t}|dG(t)| \le Ce^{-(\lambda_++\varepsilon-\tau)(b+x)}\|g\|_\tau$$

and, for $\operatorname{Re}\tau \le \lambda_+ + \varepsilon$,

$$\left|\int_{(b+x,\infty)} e^{\tau y} d\varphi_{z,s}(y)\right| = \left|\int_{(b+x,\infty)} e^{\tau u}\, d\Psi_{z,s}(u) \int_{(-\infty,0]} e^{\tau t} dG(t)\right| \le Ce^{-(\lambda_++\varepsilon-\operatorname{Re}\tau)(b+x)}|g(\tau)|.$$

The lemma is proven.

The following lemma can be proven by analogy.

Lemma 3. *Let conditions* A1 *and* A2 *hold. In addition, we assume that* $\gamma > 0$ *and* $\varphi(\gamma) > 1$ *if* $\mathbf{E}X_1 > 0$. *Then, for every function* $g(\lambda) = \int_0^\infty e^{\lambda y}\,dG(y) \in \Pi_+(0)$ *there exists* $\delta > 0$ *such that the following representation takes place for* $|z| < 1$, $s \in L_\delta$, *and* $\operatorname{Re}\lambda \ge \lambda_-$*:*

$$(L_-g)(z,s,\lambda) = U(z,s,\lambda)e^{(\lambda_-(s)-\lambda)a}g(\lambda_-(s)) + \frac{(z-s)R_-(s,\lambda)}{sR_-(z,\lambda)}\int\limits_{-\infty}^{-a} e^{\lambda y}\,d\theta_{z,s}(y),$$

where $U(z,s,\lambda)$ *is defined in* (2) *and, for all* $x \ge 0$, *we have*

$$\int\limits_{-\infty}^{-a-x} e^{\tau y}|d\theta_{z,s}(y)| \le Ce^{(\lambda_- -\varepsilon-\tau)(a+x)}\|g\|_\tau, \quad \tau \ge \lambda_- - \varepsilon,$$

$$\biggl|\int\limits_{-\infty}^{-a-x} e^{\tau y}d\theta_{z,s}(y)\biggr| \le Ce^{(\lambda_- -\varepsilon-\operatorname{Re}\tau)(a+x)}|g(\tau)|, \quad \operatorname{Re}\tau \ge \lambda_- - \varepsilon.$$

Remark. 1. The principal parts of the resultant asymptotic representations and the remainder estimates do not change if we take $[b,\infty)$ instead of (b,∞) in the definition of L_+ and $(-\infty,-a]$ instead of $(-\infty,-a)$ in the definition of L_-.

2. Easy calculations show that the remainder in Lemma 2 vanishes if $\mathbf{P}(X_1 > t) = ce^{-\alpha_1 t}$, $t > 0$. In the same way, we have $\theta_{z,s}(y) \equiv 0$ in Lemma 3 under the condition $\mathbf{P}(X_1 < t) = ce^{\alpha_2 t}$, $t < 0$.

4. Asymptotic Representations for the Moment Generating Functions in One-Sided Boundary Problems

Using the above-obtained lemmas, we turn now to finding the asymptotic representations for the functions $f_{a,b}(z,u,\lambda)$ in one-sided boundary problems.

Let $a = \infty$, $0 \le b < \infty$, and $e(z,s,\lambda) \equiv 1$. Employing (12) and Lemma 2, for $|z| < 1$, we obtain

$$\begin{aligned}(s-z)z^{-1}q_2(z,s,\lambda) &= (L_+e)(z,s,\lambda) = \frac{R_+(s,\lambda)}{R_+(z,\lambda)}\biggl[\frac{R_+(z,\lambda)}{R_+(s,\lambda)}\biggr]^{(b,\infty)} \\ &= V(z,s,\lambda)e^{(\lambda-\lambda_+(s))b} + \frac{(z-s)R_+(s,\lambda)}{sR_+(z,\lambda)}\int\limits_{(b,\infty)} e^{\lambda y}\,d\varphi^{(1)}_{z,s}(y),\end{aligned} \tag{14}$$

where

$$\int\limits_{(b+x,\infty)} e^{\tau y}\bigl|d\varphi^{(1)}_{z,s}(y)\bigr| \le Ce^{-(\lambda_+ +\varepsilon-\tau)(b+x)}, \quad \tau \le \lambda_+ + \varepsilon, \tag{15}$$

uniformly in z and $s \in L_\delta$ for $x \ge 0$.

Similarly, using (11) and Lemma 3, we have

$$\begin{aligned}(s-z)z^{-1}q_1(z,s,\lambda) &= (L_-e)(z,s,\lambda) = \frac{R_-(s,\lambda)}{R_-(z,\lambda)}\biggl[\frac{R_-(z,\lambda)}{R_-(s,\lambda)}\biggr]^{(-\infty,-a)} \\ &= U(z,s,\lambda)e^{(\lambda_-(s)-\lambda)a} + \frac{(z-s)R_-(s,\lambda)}{sR_-(z,\lambda)}\int\limits_{-\infty}^{-a} e^{\lambda y}\,d\theta^{(1)}_{z,s}(y)\end{aligned} \tag{16}$$

for $b = \infty$ and $|z| < 1$ with

$$\int\limits_{-\infty}^{-a-x} e^{\tau y}\bigl|d\theta^{(1)}_{z,s}(y)\bigr| \le Ce^{(\lambda_- -\varepsilon-\tau)(a+x)}, \quad \lambda_- - \varepsilon \le \tau, \tag{17}$$

uniformly in $s \in L_\delta$ for $x \ge 0$.

Inserting the so-obtained representations (14) and (16) into (9) and passing to the variable $v = u^{-1}$ in accord with (13), we arrive at the following assertion.

Theorem 3. *Let $a \geq 0$ and $b \geq 0$ and let conditions* A1 *and* A2 *be satisfied. Then there are $\delta > 0$ and $\varepsilon > 0$ such that the following representations take place for $|sv| < 1$, $s \in L_\delta$, and $\operatorname{Re}\lambda = 0$:*

1) *If $\beta > 0$ and, moreover, $\varphi(\beta) > 1$ for $\mathbf{E}X_1 < 0$ then*

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}(v^{T_n((b,\infty))} e^{\lambda S_n}) = \frac{1}{1 - s\varphi(\lambda)}$$
$$\times \left\{ 1 - V(sv, s, \lambda) e^{(\lambda - \lambda_+(s))b} - \frac{(v-1)R_+(s,\lambda)}{R_+(sv,\lambda)} \int\limits_{(b,\infty)} e^{\lambda y} d\varphi^{(1)}_{sv,s}(y) \right\}, \tag{18}$$

where

$$\int\limits_{(b,\infty)} \left| d\varphi^{(1)}_{sv,s}(y) \right| \leq C e^{-(\lambda_+ + \varepsilon)b}$$

uniformly in $|sv| < 1$, $s \in L_\delta$.

2) *If $\gamma > 0$ and $\varphi(\gamma) > 1$ for $\mathbf{E}X_1 > 0$ then*

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}(v^{T_n((-\infty,-a))} e^{\lambda S_n}) = \frac{1}{1 - s\varphi(\lambda)}$$
$$\times \left\{ 1 - U(sv, s, \lambda) e^{(\lambda_-(s) - \lambda)a} - \frac{(v-1)R_-(s,\lambda)}{R_-(sv,\lambda)} \int\limits_{-\infty}^{-a} e^{\lambda y} d\theta^{(1)}_{sv,s}(y) \right\}, \tag{19}$$

where

$$\int\limits_{-\infty}^{-a} \left| d\theta^{(1)}_{sv,s}(y) \right| \leq C e^{(\lambda_- - \varepsilon)a}$$

uniformly in $|sv| < 1$, $s \in L_\delta$.

Corollary 2. *Let $a \leq 0$ and $\mathbf{P}(X_1 < t) = ce^{\alpha t}$, $t < 0$. Then*

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}(v^{T_n((-\infty,-a))} e^{\lambda S_n}) = \frac{1}{1 - s\varphi(\lambda)} \left(1 - \frac{\lambda_-(s) - \lambda_-(sv)}{\lambda - \lambda_-(sv)} e^{(\lambda_-(s) - \lambda)a} \right).$$

If $b \geq 0$ and $\mathbf{P}(X_1 > t) = ce^{-\alpha t}$, $t > 0$, then

$$1 + \sum_{n=1}^{\infty} s^n \mathbf{E}(v^{T_n((b,\infty))} e^{\lambda S_n}) = \frac{1}{1 - s\varphi(\lambda)} \left(1 - \frac{\lambda_+(s) - \lambda_+(sv)}{\lambda - \lambda_+(sv)} e^{(\lambda - \lambda_+(s))b} \right).$$

This assertion ensues from Remark 2 of Item 3 and the following simple forms of the factorization components:

$$R_-(s,\lambda) = \frac{\lambda - \lambda_-(s)}{\lambda + \alpha} \quad \text{if } \mathbf{P}(X_1 < t) = ce^{\alpha t}, \ t < 0,$$
$$R_+(s,\lambda) = \frac{\lambda - \lambda_+(s)}{\lambda - \alpha} \quad \text{if } \mathbf{P}(X_1 > t) = ce^{-\alpha t}, \ t > 0.$$

Let for brevity $T_n = T_n((b,\infty))$. From (18) it follows that

$$\frac{1}{1-s} - 1 - \sum_{n=1}^{\infty} s^n \mathbf{E} v^{T_n} = \sum_{n=1}^{\infty} s^n (1 - \mathbf{E} v^{T_n})$$
$$= \frac{V(sv, s, 0) e^{-\lambda_+(s)b}}{(1-s)} + \frac{(v-1)R_+(s,0)}{(1-s)R_+(sv,0)} \int\limits_{(b,\infty)} d\varphi^{(1)}_{sv,s}(y). \tag{20}$$

Next,

$$\mathbf{E}v^{T_n} = 1 + (v-1)\mathbf{E}T_n + \frac{(v-1)^2}{2}\mathbf{E}T_n(T_n - 1) + \dots .$$

Denote

$$v(s) = -\frac{R_+(s,0)}{\lambda_+(s)R'_+(s,\lambda_+(s))}, \quad Z(s,v) = \frac{R_+(sv,\lambda_+(s))}{R_+(sv,0)}.$$

Here $V(sv,s,0) = v(s)Z(s,v)$. The function $Z(s,v)$ is analytic in v in some neighborhood of the unity for $s \in L_\delta$ and $Z(s,1) = 0$; therefore,

$$\frac{R_+(sv,\lambda_+(s))}{R_+(sv,0)} = (v-1)A(s) + \dots, \quad A(s) = \frac{\partial Z(s,v)}{\partial v}\bigg|_{v=1}.$$

Thus, if $v = 1$ then (20) implies the following representation.

Corollary 3. *Under the conditions of Theorem* 3, *there is a number* $\delta > 0$ *such that*

$$\sum_{n=1}^{\infty} s^n \mathbf{E}T_n((b,\infty)) = \frac{v(s)e^{-\lambda_+(s)b}A(s)}{s-1} + \frac{1}{s-1}\int\limits_{(b,\infty)} d\varphi^{(1)}_{s,s}(y), \quad s \in L_\delta.$$

A similar assertion can be easily stated for $\mathbf{E}T_n((-\infty,-a))$ as well.

5. Asymptotics of the Moment Generating Functions in Two-Sided Boundary Problems

Here we consider the case in which $D = [-a,b]$, $a \ge 0$, and $b \ge 0$. Suppose that the conditions of both Lemmas 2 and 3 hold simultaneously; this provides the existence of the two zeros $\lambda_\pm(s)$ for $s \in L_\delta$. The rest of the article is organized as follows: We first find the asymptotic representations for the remaining summands on the right-hand sides of (10) (the asymptotics of the first summands $(L_\pm h)(z,s,\lambda)$ was found above). Inserting these representations into (10) with the subsequent rather difficult analysis of so-obtained expressions leads to the asymptotic representations for the functions $q_2(z,s,\lambda)$ (Theorem 5) and $q_1(z,s,\lambda)$ (Theorem 6). For finding the asymptotic representation of $f_{a,b}(z,u,\lambda)$, we are left with inserting the expressions for $q_i(z,s,\lambda)$ into (9). In result, we obtain a rather bulky expression that we omit for saving room. Instead of it, we give a more compact expression for $f_{a,b}(z,u,0)$, which completes the proof of Theorem 1 at the end of this article.

So, in accordance with Lemma 3 we find

$$(L_- q_2)(z,s,\lambda) = U(z,s,\lambda)e^{(\lambda_-(s)-\lambda)a}q_2(z,s,\lambda_-(s)) + \frac{(z-s)R_-(s,\lambda)}{sR_-(z,\lambda)}\int\limits_{-\infty}^{-a} e^{\lambda y}\, d\theta^{(2)}_{z,s}(y), \tag{21}$$

where, for $x \ge 0$, the relations

$$\int\limits_{-\infty}^{-a-x} e^{\tau y}\big|d\theta^{(2)}_{z,s}(y)\big| \le Ce^{(\lambda_- - \varepsilon - \tau)(a+x)}\|q_2(z,s,.)\|_\tau, \quad \lambda_- - \varepsilon \le \tau,$$

$$\bigg|\int\limits_{-\infty}^{-a-x} e^{\tau y}\, d\theta^{(2)}_{z,s}(y)\bigg| \le Ce^{(\lambda_- - \varepsilon - \operatorname{Re}\tau)(a+x)}|q_2(z,s,\tau)|, \quad \lambda_- - \varepsilon \le \operatorname{Re}\tau, \tag{22}$$

hold uniformly in $|z| < 1$ and $s \in L_\delta$. Using Lemma 2 and (16) and recalling that $\mu(s) = e^{\lambda_-(s)-\lambda_+(s)}$, we obtain

$$(L_+L_-h)(z,s,\lambda) = V(z,s,\lambda)e^{(\lambda-\lambda_+(s))b}\bigg\{\frac{zU(z,s,\lambda_+(s))\mu^a(s)}{(s-z)}$$

$$-\frac{zR_-(s,\lambda_+(s))}{sR_-(z,\lambda_+(s))}\int\limits_{-\infty}^{-a} e^{\lambda_+(s)y}\,d\theta^{(1)}_{z,s}(y)\bigg\} + \frac{(z-s)R_+(s,\lambda)}{sR_+(z,\lambda)}\int\limits_{(b,\infty)} e^{\lambda y}\,d\varphi^{(2)}_{z,s}(y), \tag{23}$$

where, for $x \ge 0$,

$$\int\limits_{(b+x,\infty)} e^{\tau y}\big|d\varphi^{(2)}_{z,s}(y)\big| \le Ce^{-(\lambda_+ +\varepsilon-\tau)(b+x)}\|(L_-h)(z,s,.)\|_\tau, \quad \tau \le \lambda_+ + \varepsilon,$$

$$\bigg|\int\limits_{(b+x,\infty)} e^{\tau y}\,d\varphi^{(2)}_{z,s}(y)\bigg| \le Ce^{-(\lambda_+ +\varepsilon-\operatorname{Re}\tau)(b+x)}|(L_-h)(z,s,\tau)|, \quad \operatorname{Re}\tau \le \lambda_+ + \varepsilon,$$

uniformly in $s \in L_\delta$ and $|z| < 1$.

Employing Lemma 2 and (21), we find

$$(L_+L_-q_2)(z,s,\lambda) = V(z,s,\lambda)e^{(\lambda-\lambda_+(s))b}$$

$$\times\bigg\{U(z,s,\lambda_+(s))q_2(z,s,\lambda_-(s))\mu^a(s) + \frac{(z-s)R_-(s,\lambda_+(s))}{sR_-(z,\lambda_+(s))}\int\limits_{-\infty}^{-a} e^{\lambda_+(s)y}\,d\theta^{(2)}_{z,s}(y)\bigg\}$$

$$+\frac{(z-s)R_+(s,\lambda)}{sR_+(z,\lambda)}\int\limits_{(b,\infty)} e^{\lambda y}\,d\varphi^{(3)}_{z,s}(y); \tag{24}$$

here

$$\int\limits_{(b+x,\infty)} e^{\tau y}\big|d\varphi^{(3)}_{z,s}(y)\big| \le Ce^{-(\lambda_+ +\varepsilon-\tau)(b+x)}\|(L_-q_2)(z,s,.)\|_\tau, \quad \tau \le \lambda_+ + \varepsilon,$$

$$\bigg|\int\limits_{(b+x,\infty)} e^{\tau y}\,d\varphi^{(3)}_{z,s}(y)\bigg| \le Ce^{-(\lambda_+ +\varepsilon-\operatorname{Re}\tau)(b+x)}|(L_-q_2)(z,s,\tau)|, \quad \operatorname{Re}\tau \le \lambda_+ + \varepsilon,$$

uniformly in $|z| < 1$ and $s \in L_\delta$ for $x \ge 0$.

Inserting (14), (23), and (24) into (10) and using (2), we obtain

$$q_2(z,s,\lambda) = \frac{zV(z,s,\lambda)e^{(\lambda-\lambda_+(s))b}}{(s-z)}$$

$$\times\bigg\{1 - H_1(z,s)\mu^a(s)\Big(1 - \frac{s-z}{z}q_2(z,s,\lambda_-(s))\Big) + \frac{s-z}{z}\Delta_1(z,s)\bigg\} + \Delta_2(z,s,\lambda), \tag{25}$$

where

$$\Delta_1(z,s) = \frac{R_-(s,\lambda_+(s))}{R_-(z,\lambda_+(s))}\bigg(\frac{z}{s}\int\limits_{-\infty}^{-a} e^{\lambda_+(s)y}\,d\theta^{(1)}_{z,s}(y) + \frac{z-s}{s}\int\limits_{-\infty}^{-a} e^{\lambda_+(s)y}d\theta^{(2)}_{z,s}(y)\bigg), \tag{26}$$

$$\Delta_2(z,s,\lambda) = \frac{R_+(s,\lambda)}{R_+(z,\lambda)}\bigg(\frac{z-s}{s}\int\limits_{(b,\infty)} e^{\lambda y}\,d\big(\varphi^{(3)}_{z,s}(y) - \varphi^{(2)}_{z,s}(y)\big) - \frac{z}{s}\int\limits_{(b,\infty)} e^{\lambda y}\,d\varphi^{(1)}_{z,s}(y)\bigg).$$

To find $q_2(z,s,\lambda_-(s))$, we put $\lambda=\lambda_-(s)$ in (25). Using (2) for the functions H, H_1, and H_2, we deduce

$$q_2(z,s,\lambda_-(s))(1-H(z,s)\mu^{b+a}(s))$$
$$=z(s-z)^{-1}H_2(z,s)\mu^b(s)\bigg(1-H_1(z,s)\mu^a(s)+\frac{s-z}{z}\Delta_1(z,s)\bigg)+\Delta_2(z,s,\lambda_-(s)),$$
$$1-(s-z)z^{-1}q_2(z,s,\lambda_-(s))$$
$$=1-\frac{H_2(z,s)\mu^b(s)\big(1-H_1(z,s)\mu^a(s)+\frac{s-z}{z}\Delta_1(z,s)\big)+\frac{s-z}{z}\Delta_2(z,s,\lambda_-(s))}{1-H(z,s)\mu^{b+a}(s)}$$
$$=\frac{1-H_2(z,s)\mu^b(s)\big(1+\frac{s-z}{z}\Delta_1(z,s)\big)+\frac{s-z}{z}\Delta_2(z,s,\lambda_-(s))}{1-H(z,s)\mu^{b+a}(s)},$$

and, finally,

$$q_2(z,s,\lambda)=\frac{zV(z,s,\lambda)e^{(\lambda-\lambda_+(s))b}}{(s-z)}$$
$$\times\bigg\{1-H_1(z,s)\mu^a(s)\frac{1-H_2(z,s)\mu^b(s)\big(1+\frac{s-z}{z}\Delta_1(z,s)\big)+\frac{s-z}{z}\Delta_2(z,s,\lambda_-(s))}{1-H(z,s)\mu^{b+a}(s)}$$
$$+\frac{s-z}{z}\Delta_1(z,s)\bigg\}+\Delta_2(z,s,\lambda)=\frac{zV(z,s,\lambda)e^{(\lambda-\lambda_+(s))b}}{(s-z)}\bigg\{\frac{1-H_1(z,s)\mu^a(s)}{1-H(z,s)\mu^{b+a}(s)}$$
$$+\frac{(s-z)(H(z,s)\mu^{b+a}(s)\Delta_1(z,s)-H_1(z,s)\mu^a(s)\Delta_2(z,s,\lambda_-(s)))}{z(1-H(z,s)\mu^{b+a}(s))}+\frac{s-z}{z}\Delta_1(z,s)\bigg\}$$
$$+\Delta_2(z,s,\lambda)=\frac{zV(z,s,\lambda)e^{(\lambda-\lambda_+(s))b}}{(s-z)}\bigg\{\frac{1-H_1(z,s)\mu^a(s)}{1-H(z,s)\mu^{b+a}(s)}+\Delta_3(z,s)\bigg\}+\Delta_2(z,s,\lambda),$$

where

$$\Delta_3(z,s)=\frac{s-z}{z}\frac{\Delta_1(z,s)-\Delta_2(z,s,\lambda_-(s))H_1(z,s)\mu^a(s)}{1-H(z,s)\mu^{b+a}(s)}.$$

Recalling that, in view of (4), the remainders in the assertions of Lemmas 2 and 3 are absent, we conclude that $\Delta_1(z,s)$ and $\Delta_2(z,s,\lambda)$ vanish in this particular case. Using (5)–(7), we obtain

$$q_2(z,s,\lambda)=\frac{z(\lambda_+(s)-\alpha_1)(\lambda_+(s)+\alpha_2)}{s(\lambda-\lambda_+(z))(\lambda_+(s)-\lambda_-(z))}\frac{1-H_1(z,s)\mu^a(s)}{1-H(z,s)\mu^{b+a}(s)}e^{(\lambda-\lambda_+(s))b}.$$

The right-hand side of this expression can be simplified, since

$$\frac{(\lambda_+(s)-\lambda_+(z))(\lambda_+(s)-\lambda_-(z))}{(\lambda_+(s)-\alpha_1)(\lambda_+(s)+\alpha_2)}=1-z\varphi(\lambda_+(s))=1-\frac{z}{s}.$$

In a similar manner, we can calculate $q_1(z,s,\lambda)$ for this situation. We thus arrive at the following assertion (see also [13]).

Theorem 4. *Let condition* (4) *hold. Then*

$$q_1(z,s,\lambda)=\frac{z(\lambda_-(s)-\lambda_-(z))}{(s-z)(\lambda-\lambda_-(z))}\frac{1-H_2(z,s)\mu^b(s)}{1-H(z,s)\mu^{b+a}(s)}e^{(\lambda_-(s)-\lambda)a},$$

$$q_2(z,s,\lambda)=\frac{z(\lambda_+(s)-\lambda_+(z))}{(s-z)(\lambda-\lambda_+(z))}\frac{1-H_1(z,s)\mu^a(s)}{1-H(z,s)\mu^{b+a}(s)}e^{(\lambda-\lambda_+(s))b}$$

for $a\geq 0$ *and* $b\geq 0$.

The expressions for $q_i(z,s,\lambda)$ can be easily inverted in the variable λ.

Corollary 4. *Under the conditions of Theorem 4, for every $x \ge 0$ we have*

$$\sum_{n=1}^{\infty} z^n \sum_{k=0}^{n} u^k \mathbf{P}(T_n = k, S_n \ge b + x)$$
$$= \frac{\lambda_+(s) - \lambda_+(z)}{(1-u)\lambda_+(z)} \frac{1 - H_1(z,s)\mu^a(s)}{1 - H(z,s)\mu^{b+a}(s)} e^{-\lambda_+(s)b - \lambda_+(z)x}, \quad s = zu.$$

We could further apply the identity (9) and use the equality $\lambda_+(s)\lambda_-(s) = \alpha_1\alpha_2(s-1)$ (this follows from Vièta's theorem). We finally obtain the following:

Corollary 5. *Under the conditions of Theorem 4, we have*

$$1 + f_{a,b}(z,u,\lambda) = \frac{1}{1 - s\varphi(\lambda)} \left\{ 1 - \frac{\lambda_+(s) - \lambda_+(z)}{\lambda - \lambda_+(z)} \frac{1 - H_1(z,s)\mu^a(s)}{1 - H(z,s)\mu^{b+a}(s)} e^{(\lambda - \lambda_+(s))b} \right\}$$
$$- \frac{1}{1 - s\varphi(\lambda)} \left\{ \frac{\lambda_-(s) - \lambda_-(z)}{\lambda - \lambda_-(z)} \frac{1 - H_2(z,s)\mu^b(s)}{1 - H(z,s)\mu^{b+a}(s)} e^{(\lambda_-(s) - \lambda)a} \right\}, \quad s = zu.$$

We turn now to estimating the remainders Δ_i in the general case. We will do this on assuming that $\mathbf{E}\, X_1 = 0$. In this case $\lambda_\pm = 0$, and the following expansions take place in a neighborhood of the unity with a cut along the ray $z \ge 1$ (see [7]):

$$\lambda_\pm(z) = \pm\psi_1(1-z)^{1/2} + \psi_2(1-z) + \cdots$$

(throughout the sequel, we bear in mind the principal value of the square root), where $\psi_1 = \sqrt{2/\sigma^2}$, $\psi_2 = \mu_3/(3\sigma^4)$, $\mu_k = E\, X_1^k$, and $\sigma^2 = \mu_2 - \mu_1^2$.

In what follows, we assume that $z \in L_\delta$ alongside $s \in L_\delta$ for small $\delta > 0$; moreover, $|z| < |s|$.

We first estimate $\Delta_1(z,s)$. Observe that

$$\left| \frac{R_-(s, \lambda_+(s))}{R_-(z, \lambda_+(s))} \right| \le \left| \frac{R_-(s, \lambda_+(s))}{\lambda_+(s) - \lambda_-(s)} \right| \left| \frac{\lambda_+(s) - \lambda_-(z)}{R_-(z, \lambda_+(s))} \right| \left| \frac{\lambda_+(s) - \lambda_-(s)}{\lambda_+(s) - \lambda_-(z)} \right|$$
$$\le \frac{C|1-s|^{1/2}}{|(1-s)^{1/2} + (1-z)^{1/2}|} \le C_1.$$

Therefore,

$$|\Delta_1(z,s)| \le C_1 \left| \int_{-\infty}^{-a} e^{\lambda_+(s)y} d\theta_{z,s}^{(1)}(y) \right| + C_2 \left| \frac{z-s}{s} \int_{-\infty}^{-a} e^{\lambda_+(s)y}\, d\theta_{z,s}^{(2)}(y) \right|.$$

Given small δ, we have $0 < \operatorname{Re}\lambda_+(s) < \varepsilon/2$; hence, by (17) and (22),

$$\left| \int_{-\infty}^{-a} e^{\lambda_+(s)y}\, d\theta_{z,s}^{(1)}(y) \right| \le Ce^{-(\varepsilon - \operatorname{Re}\lambda_+(s))a} \le Ce^{-\varepsilon a/2}, \tag{27}$$

$$\left| \int_{-\infty}^{-a} e^{\lambda_+(s)y} d\theta_{z,s}^{(2)}(y) \right| \le Ce^{-\varepsilon a/2} |q_2(z,s,\lambda_+(s))|. \tag{28}$$

The function $q_2(z,s,\lambda)$ is determined for $\operatorname{Re}\lambda \le 0$ and analytic in the domain $\operatorname{Re}\lambda < 0$. Expressing Q_2 from (8), we obtain

$$q_2(z,s,\lambda) = Q_2(z,u,\lambda) = \frac{1 - (1 + Q_0(z,u,\lambda))(1 - zu\varphi(\lambda))}{u(1 - z\varphi(\lambda))} - Q_1(z,u,\lambda), \tag{29}$$

from which we see that $q_2(z,s,\lambda)$ admit analytic continuation to the domain $\operatorname{Re}\lambda < \operatorname{Re}\lambda_+(z)$. If $|z|<|s|$ then $\operatorname{Re}\lambda_+(s) < \operatorname{Re}\lambda_+(z)$, which justifies the use of $q_2(z,s,\lambda_+(s))$ on the right-hand side of (28).

Putting $\lambda=\lambda_\pm(s)$ in (29), for $|z|<|s|$, we infer

$$q_2(z,s,\lambda_\pm(s)) + q_1(z,s,\lambda_\pm(s)) = \frac{z}{s-z}. \tag{30}$$

This relation is of interest in its own right and, in its form, looks like one of the well-known Wald identities.

Next, let $\tilde{q}_2(z,s,\lambda)$ coincide with $q_2(z,s,\lambda)$ for $a=\infty$. Then, clearly,

$$\tilde{q}_2(z,s,\lambda_\pm(s)) = \frac{z}{s-z}$$

and, provided that $z\in(1-\delta,1)$ and $s\in(z,1)$ are real,

$$\left|\frac{s-z}{z} q_2(z,s,\lambda_\pm(s))\right| \le \left|\frac{s-z}{z}\tilde{q}_2(z,s,\lambda_\pm(s))\right| = 1.$$

Hence, for $z\in L_\delta$, $s\in L_\delta$, and $|z|<|s|$, with δ sufficiently small, we have

$$|(s-z)q_2(z,s,\lambda_\pm(s))| \le 2. \tag{31}$$

This estimate can be obtained on using the above representations for $(L_+h)(z,s,\lambda_+(s))$, $(L_+L_-h)(z,s,\lambda_+(s))$, and $(L_+L_-q_2)(z,s,\lambda_+(s))$.

Together with (27) and (28), this yields the following estimate for these z and s:

$$|\Delta_1(z,s)| \le Ce^{-\varepsilon a/2}.$$

We now estimate the quantity

$$\Delta_2(z,s,\lambda_-(s)) = \frac{R_+(s,\lambda_-(s))}{R_+(z,\lambda_-(s))}\biggl(\frac{z-s}{s}\int\limits_{(b,\infty)} e^{\lambda_-(s)y}\,d\bigl(\varphi^{(3)}_{z,s}(y)-\varphi^{(2)}_{z,s}(y)\bigr) - \frac{z}{s}\int\limits_{(b,\infty)} e^{\lambda_-(s)y}\,d\varphi^{(1)}_{z,s}(y)\biggr).$$

As above, we establish that

$$\left|\frac{R_+(s,\lambda_-(s))}{R_+(z,\lambda_-(s))}\right| \le C$$

and, by (15),

$$\biggl|\int\limits_{(b,\infty)} e^{\lambda_-(s)y}\,d\varphi^{(1)}_{z,s}(y)\biggr| \le Ce^{-(\varepsilon-\operatorname{Re}\lambda_-(s))b} \le Ce^{-\varepsilon b}.$$

Moreover,

$$\biggl|\int\limits_{(b,\infty)} e^{\lambda_-(s)y}\,d\bigl(\varphi^{(3)}_{z,s}(y)-\varphi^{(2)}_{z,s}(y)\bigr)\biggr| \le \biggl|\int\limits_{(b,\infty)} e^{\lambda_-(s)y}\,d\varphi^{(2)}_{z,s}(y)\biggr|$$
$$+\biggl|\int\limits_{(b,\infty)} e^{\lambda_-(s)y}\,d\varphi^{(3)}_{z,s}(y)\biggr| \le e^{-(\varepsilon-\operatorname{Re}\lambda_-(s)b}(|(L_-h)(z,s,\lambda_-(s))|$$
$$+|(L_-q_2)(z,s,\lambda_-(s))|) \le e^{-\varepsilon b}(|(L_-h)(z,s,\lambda_-(s))| + |(L_-q_2)(z,s,\lambda_-(s))|).$$

Employ the relations

$$(L_-h)(z,s,\lambda_-(s)) = \frac{z}{s-z}, \quad (L_-q_2)(z,s,\lambda_-(s)) = q_2(z,s,\lambda_-(s))$$

which are immediate from (16) and (21) (the first of them also follows from (29) if $b=\infty$, the second ensues from the comparison of (10) and (30)). Considering (31), we conclude that

$$|\Delta_2(z,s,\lambda_-(s))| \le Ce^{-\varepsilon b}(1+|(s-z)q_2(z,s,\lambda_-(s))|) \le C_1 e^{-\varepsilon b}.$$

Finally, show that

$$\frac{1}{|1-H(z,s)\mu^{b+a}(s)|} \le C < \infty \tag{32}$$

for $s \in L_\delta$, $z \in L_\delta$, and $|z|<|s|$. Recall (see (2)) that

$$H(z,s) = a_-(s)a_+(s)\frac{R_-(z,\lambda_-(s))}{R_-(z,\lambda_+(s))}\,\frac{R_+(z,\lambda_+(s))}{R_+(z,\lambda_-(s))},$$

where

$$a_-(s) = \frac{R_-(s,\lambda_+(s))}{(\lambda_+(s)-\lambda_-(s))R'_-(s,\lambda_-(s))}, \quad a_+(s) = \frac{R_+(s,\lambda_-(s))}{(\lambda_-(s)-\lambda_+(s))R'_+(s,\lambda_+(s))}.$$

Clearly, $|\mu(s)| \le 1$ and $a_-(s)a_+(s) = 1+O(\sqrt{1-s})$ as $s \to 1$. Further, the quantity $\frac{R_-(z,\lambda_-(s))}{R_-(z,\lambda_+(s))}\frac{R_+(z,\lambda_+(s))}{R_+(z,\lambda_-(s))}$ vanishes for $s=z$; hence, its absolute value is at most 1/2 if s and z are sufficiently close. This implies (32).

The remainder $\Delta_2(z,s,\lambda)$ has the form

$$\Delta_2(z,s,\lambda) = \frac{R_+(s,\lambda)}{R_+(z,\lambda)}\int_b^\infty e^{\lambda y}\, d\varphi_{z,s}(y).$$

Show that $|\Delta_2(z,s,0)| \le Ce^{-\varepsilon b}$. Note that

$$\left|\frac{R_+(s,0)}{R_+(z,0)}\right| = \left|\frac{-\lambda_+(s)R'_+(s,\lambda_+(s))+\dots}{-\lambda_+(z)R'_+(z,\lambda_+(z)0)+\dots}\right| = \left|\frac{\sqrt{1-s}}{\sqrt{1-z}}\left(\frac{R'_+(s,\lambda_+(s))}{R'_+(z,\lambda_+(z))}+\dots\right)\right| \le C.$$

In view of (26), we infer

$$\begin{aligned}\left|\int_b^\infty d\varphi_{z,s}(y)\right| &\le \left|\frac{z}{s}\int_b^\infty d\varphi^{(1)}_{z,s}(y)\right| + \left|\frac{z-s}{s}\int_b^\infty \left(d\varphi^{(2)}_{z,s}(y)+d\varphi^{(3)}_{z,s}(y)\right)\right| \\ &\le C_1e^{-\varepsilon b} + C_2|s-z|e^{-\varepsilon b}(|(L_-h)(z,s,0)|+|(L_-q_2)(z,s,0)|).\end{aligned}$$

By (16) and (21), we have

$$|(s-z)(L_-h)(z,s,0)| \le \left|\frac{a_-(s)\,e^{\lambda_-(s)a}R_-(z,\lambda_-(s))}{R_-(z,0)}\right| + \left|\frac{(z-s)R_-(s,0)}{sR_-(z,0)}\int_{-\infty}^{-a} d\theta^{(1)}_{z,s}(y)\right| \le C,$$

since

$$\begin{aligned}\left|\frac{R_-(z,\lambda_-(s))}{R_-(z,0)}\right| &= \left|\frac{(\lambda_-(s)-\lambda_-(z))R'_-(z,\lambda_-(z))+\dots}{-\lambda_-(z)R'_-(z,\lambda_-(z))+\dots}\right| \\ &= \left|\frac{\left(\sqrt{1-s}-\sqrt{1-z}\right)R'_-(z,\lambda_-(z))+\dots}{\sqrt{1-z}R'_-(z,\lambda_-(z))+\dots}\right| \le C,\end{aligned}$$

and

$$\begin{aligned}|(s-z)(L_-q_2)(z,s,0)| &\le \left|\frac{a_-(s)e^{\lambda_-(s)a}R_-(z,\lambda_-(s))(s-z)q_2(z,\lambda_-(s))}{R_-(z,0)}\right| \\ &+\left|\frac{(s-z)^2R_-(s,0)}{sR_-(z,0)}\int_{-\infty}^{-a} d\theta^{(2)}_{z,s}(y)\right| \le C_1 + C_2e^{-\varepsilon a}|(s-z)^2q_2(z,s,0)|.\end{aligned}$$

Let, as above, $\tilde{q}_2(z,s,0)$ coincide with $q_2(z,s,0)$ for $a=\infty$. If $z \in (1-\delta,1)$ and $s \in (z,1)$ are real then

$$\left|\frac{s-z}{z} q_2(z,s,0)\right| \le \left|\frac{s-z}{z}\ \tilde{q}_2(z,s,0)\right| \le \left|\frac{s-z}{z}\ \tilde{q}_2(z,s,\lambda_+(s))\right| \le \frac{1}{|z|}.$$

Hence, for $z \in L_\delta$, $s \in L_\delta$, and $|z| < |s|$, with δ sufficiently small, we have the estimate

$$|(s-z)q_2(z,s,0)| \le 2.$$

We thus proved the assertion to appear below. Before formulating it, we recall that the quantities $V(z,s,\lambda)$, $U(z,s,\lambda)$, $H(z,s)$, $H_i(z,s)$, and $\mu(s)$ were introduced before Theorem 1 and the functions $q_i(z,s,\lambda)$, before Theorem 2.

Theorem 5. *Let* $\mathbf{E}X_1 = 0$, $a \ge 0$, *and* $b \ge 0$ *and let conditions* A1 *and* A2 *be satisfied with* $\gamma > 0$ *and* $\beta > 0$. *Then there are* $\delta > 0$ *and* $\varepsilon > 0$ *such that the following representation takes place for* $z \in L_\delta$, $s \in L_\delta$, *and* $\operatorname{Re}\lambda \le 0$:

$$q_2(z,s,\lambda) = \frac{zV(z,s,\lambda)\, e^{(\lambda-\lambda_+(s))b}}{(s-z)}\left(\frac{1-H_1(z,s)\mu^a(s)}{1-H(z,s)\mu^{b+a}(s)} + (s-z)\varepsilon_1(z,s)\right) + \Delta(z,s,\lambda),$$

where $|\varepsilon_1(z,s)| \le C_1 e^{-\varepsilon a} + C_2 e^{-\varepsilon b}$ *for* $|z| < |s|$ *and the function* $\Delta(z,s,\lambda)$ *has the form*

$$\Delta(z,s,\lambda) = \frac{R_+(s,\lambda)}{R_+(z,\lambda)} \int\limits_{(b,\infty)} e^{\lambda y}\, d\varphi_{z,s}(y), \quad |\Delta(z,s,0)| \le Ce^{-\varepsilon b}.$$

Using symmetric arguments, we can prove the following

Theorem 6. *Let* $\mathbf{E}X_1 = 0$, $a \ge 0$, *and* $b \ge 0$ *and let conditions* A1 *and* A2 *be satisfied with* $\gamma > 0$ *and* $\beta > 0$. *Then there are* $\delta > 0$ *and* $\varepsilon > 0$ *such that the following representation takes place for* $z \in L_\delta$, $s \in L_\delta$, *and* $\operatorname{Re}\lambda \ge 0$:

$$q_1(z,s,\lambda) = \frac{zU(z,s,\lambda)\, e^{(\lambda_-(s)-\lambda)a}}{(s-z)}\left(\frac{1-H_2(z,s)\mu^b(s)}{1-H(z,s)\mu^{b+a}(s)} + (s-z)\varepsilon_2(z,s)\right) + \widetilde{\Delta}(z,s,\lambda),$$

where $|\varepsilon_2(z,s)| \le C_1 e^{-\varepsilon a} + C_2 e^{-\varepsilon b}$ *for* $|z| < |s|$ *and the function* $\widetilde{\Delta}(z,s,\lambda)$ *has the form*

$$\widetilde{\Delta}(z,s,\lambda) = \frac{R_-(s,\lambda)}{R_-(z,\lambda)} \int\limits_{-\infty}^{-a} e^{\lambda y}\, d\theta_{z,s}(y), \quad |\widetilde{\Delta}(z,s,0)| \le Ce^{-\varepsilon a}.$$

Recall the above-proven identity (see (9))

$$1 + f_{a,b}(z,u,\lambda) = \frac{1}{1-s\varphi(\lambda)}(1 + (1-u)(q_1(z,s,\lambda) + q_2(z,s,\lambda))).$$

Inserting the asymptotic representations of Theorems 5 and 6 for the functions q_1 and q_2 in the right-hand side of this identity, we arrive at the corresponding representation for $f_{a,b}(z,u,\lambda)$. The corollary to this assertion with the condition $\lambda = 0$ was given in the beginning of the article (Theorem 1).

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