

1 Markov Chains, Random Walks and Regenerative Processes

1.1 Markov Chains on a real line $\mathbf{R}$. Main definitions

Definition 1 Let $\{X_n, n \geq 0\}$ be a sequence of r.v.'s. We say that $X \equiv \{X_n, n \geq 0\}$ is a Markov Chain (M.C.) with initial distribution $P^{(0)}$ and with transition probabilities $\{P_n\}_{n \geq 0}$, if

$$\mathbf{P}(X_0 \in B) = P^{(0)}(B), \quad B \in \mathcal{B}$$

and $P_n : \mathbf{R} \times \mathcal{B} \rightarrow [0, 1]$ are such that

- (a) $\forall B \in \mathcal{B}, \quad P_n(\cdot, B)$ is a measurable function $P_n(\cdot, B) : \mathbf{R} \rightarrow [0, 1]$;
- (b) $\forall x \in \mathbf{R}, \quad P_n(x, \cdot)$ is a probability measure on $\mathbf{R}$;
- (c) $\forall n \geq 0, \forall B \in \mathcal{B}, \forall x_0, \dots, x_n \in \mathbf{R},$

$$\mathbf{P}(X_{n+1} \in B \mid X_0 = x_0, \dots, X_n = x_n) = \mathbf{P}(X_{n+1} \in B \mid X_n = x_n) = P_n(x_n, B).$$

Definition 2 MC is homogeneous in time, if $P_0 = P_1 = \dots = P_n = \dots \equiv P$.

We shall consider *homogeneous in time* MC's only.

Denote by $\xi_n = X_n - X_{n-1}, n \geq 1$, the increments of MC. Then ξ_n have the following conditional distributions:

for $n \geq 1, B \in \mathcal{B}$

$$\begin{aligned} \mathbf{P}(\xi_{n+1} \in B \mid X_0 = x_0, \dots, X_n = x_n) &= \mathbf{P}(X_{n+1} - x_n \in B \mid X_n = x_n) = \\ &= \mathbf{P}(X_{n+1} \in x_n + B \mid X_n = x_n) \equiv P_n(x_n, x_n + B), \end{aligned}$$

where $x + B \stackrel{\text{def}}{=} \{x + y, y \in B\}$.

Definition 3 MC $X = \{X_n, n \geq 0\}$ is a random walk (RW), if the transition probabilities $P(x, x + B)$ do not depend on x ($\forall B \in \mathcal{B}$).

For a RW, $\{\xi_n\}_{n \geq 1}$ form an i.i.d. sequence with distribution

$$\mathbf{P}(\xi_1 \in B) = P(0, B)$$

and $\xi_0 \equiv X_0 \perp \{\xi_n\}_{n \geq 1}$.

Definition 4 *MC is finite (countable), if there exists a finite (countable) set $\mathcal{X} \subseteq \mathbf{R}$:*

- (a) $P^{(0)}(\mathcal{X}) \equiv \mathbf{P}(X_0 \in \mathcal{X}) = 1$;
- (b) $\forall x \in \mathcal{X}, \quad P(x, \mathcal{X}) = 1$.

Remark 1 *Generalization onto $\mathbf{R}^d$...*

Examples.

(1) Simple RW.

$$X_0 = 0 \text{ a.s.}, \{\xi_n\} \text{ are i.i.d.}, \quad \begin{array}{c|c|c} \xi_1 & 1 & -1 \\ \hline & p & q = 1 - p \end{array}$$

Lemma 1 $\mathbf{P}(\exists n \geq 1 : X_n = 0) = 1 \iff p = q = 1/2$.

(2) Reflected RW.

- (a) reflected at point 0: $\begin{cases} X_0 = c = \text{const} \geq 0, \{\xi_n\} \text{ — i.i.d.}, \\ X_{n+1} = \max(0, X_n + \xi_{n+1}) \equiv (X_n + \xi_{n+1})^+. \end{cases}$
- (b) reflected at point N : $\begin{cases} X_{n+1} = \max(N, X_n + \xi_{n+1}) \equiv N + (X_n + \xi_{n+1} - N)^+. \\ (X_n = X'_n + N, X'_n \text{ — reflected at } 0). \end{cases}$

(3) $X_0 = c = \text{const}, X_{n+1} = |X_n + \xi_{n+1}|$.

(4) Oscillating RW.

$\{\xi_n\}, \{\eta_n\}$ are i.i.d. and mutually independent,

$$X_0 = c = \text{const}, \quad X_{n+1} = X_n + \begin{cases} \xi_n, & \text{if } X_n < 0 \\ \eta_n, & \text{if } X_n \geq 0 \end{cases}$$

1.2 Random Walks. Classification

$$X_0 = S_0 = 0, \quad X_n = S_n = \xi_1 + \dots + \xi_n, \quad n \geq 1.$$

Assumption. $\mathbf{P}(\xi_1 > 0) > 0, \quad \mathbf{P}(\xi_1 < 0) > 0.$

Set

$\overline{S}_n = \max(0, S_1, \dots, S_n)$ — partial maximum;

$\underline{S}_n = \min(0, S_1, \dots, S_n)$ — partial minimum.

$\overline{S} = \sup_{n \geq 0} S_n \equiv \lim_{n \rightarrow \infty} \overline{S}_n$ — total maximum;

$\underline{S} = \inf_{n \geq 0} S_n \equiv \lim_{n \rightarrow \infty} \underline{S}_n$ — total minimum.

$$\tau_+^S = \begin{cases} \min\{n \geq 1 : S_n > 0\}; \\ \infty, \text{ if } S_n \leq 0 \forall n \geq 1 \end{cases}$$

— the first *strict* ascending ladder epoch, or the first entrance time to $(0, \infty)$, or the first hitting time.

$$\tau_+^W = \begin{cases} \min\{n \geq 1 : S_n \geq 0\}; \\ \infty, \text{ if } S_n < 0 \forall n \geq 1 \end{cases}$$

.... *weak* ... $[0, \infty)$..

$$\tau_-^S = \begin{cases} \min\{n \geq 1 : S_n < 0\}; \\ \infty, \text{ if } S_n \geq 0 \forall n \geq 1. \end{cases}$$

$$\tau_-^W = \begin{cases} \min\{n \geq 1 : S_n \leq 0\}; \\ \infty, \text{ if } S_n > 0 \forall n \geq 1. \end{cases}$$

Define

$u_+^S = S_{\tau_+^S}$ — the first strict ascending ladder height ,

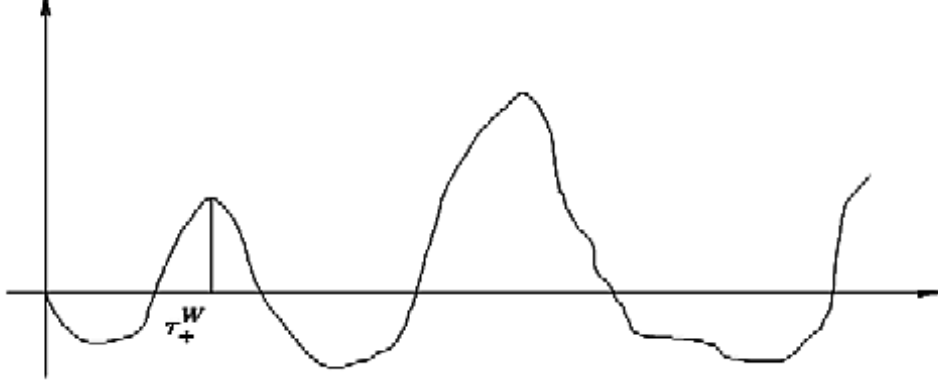
$u_+^W = S_{\tau_+^W}$ — ... strict descending ...,

$u_-^S = S_{\tau_-^S}$ — ... weak ascending ...,

$u_-^W = S_{\tau_-^W}$ — ... weak descending ...

(assume, f.e., $u_+^S = -\infty$, if $\tau_+^S = \infty$).

(?) How to check, whether $\mathbf{P}(\tau_+^W < \infty) = 1$ or < 1 ?



Define recursively: $\{\xi_n^{(1)}\} = \{\xi_n\}$, $\mu^{(1)} = \tau_+^W$, $u^{(1)} = u_+^W$.
 On event $\{\mu^{(1)} < \infty\}$: $\{\xi_n^{(2)}\} = \{\xi_{\mu^{(1)}+n}^{(1)}\}$, $\mu^{(2)}$, $u^{(2)}$.
 $\vdots$
 On event $\{\mu^{(1)} < \infty, \dots, \mu^{(m)} < \infty\}$: $\{\xi_n^{(m+1)}\} = \{\xi_{\mu^{(m)}+n}^{(m)}\}$, $\mu^{(m+1)}$, $u^{(m+1)}$.
 Then: $\mu^{(1)}, \mu^{(2)}, \dots$ are i.i.d.

$$\begin{aligned} H_\mu(n) &= \mathbf{E}\eta_\mu(n) = 1 + \sum_1^\infty \mathbf{P}(\mu^{(1)} + \dots + \mu^{(m)} \leq n) \\ &= 1 + \sum_1^n \mathbf{P}(S_n \geq 0, S_n \geq S_1, \dots, S_n \geq S_{n-1}). \end{aligned}$$

So, if $H_\mu(n) \rightarrow \infty$ as $n \rightarrow \infty$ $\implies \mathbf{P}(\mu_1 < \infty) = 1$.
 $\dots$ bounded $\implies \mathbf{P}(\mu_1 < \infty) < 1$.

(??) Assume, $\mathbf{P}(\tau_+^W < \infty) = 1$. How to check, whether $\mathbf{E}\tau_+^W < \infty$ or $= \infty$?

We know:

$$\frac{H_\mu(n)}{n} \rightarrow \frac{1}{\mathbf{E}\tau_+^W}.$$

Put $h_n = \mathbf{P}(S_n \geq 0, S_n \geq S_1, \dots, S_n \geq S_{n-1})$.

Lemma 2 $h_n = \mathbf{P}(S_1 \geq 0, S_2 \geq 0, \dots, S_n \geq 0)$.

Proof. Set $\xi'_1 = \xi_n, \xi'_2 = \xi_{n-1}, \dots, \xi'_n = \xi_1, S'_j = \xi'_1 + \dots + \xi'_j, j = 1, \dots, n$.

Note: $\{\xi'_i\}$ are i.i.d., $\xi'_1 \stackrel{D}{=} \xi_1$ and $S_n - S_{n-j} = S'_j$. Therefore,

$$\begin{aligned} h_n &= \mathbf{P}(S_n \geq 0, S_n - S_1 \geq 0, \dots, S_n - S_{n-1} \geq 0) = \\ &= \mathbf{P}(S'_n \geq 0, S'_{n-1} \geq 0, \dots, S'_1 \geq 0) = \\ &= \mathbf{P}(S_n \geq 0, S_{n-1} \geq 0, \dots, S_1 \geq 0). \end{aligned}$$

□

Corollary 1 $h_n \geq h_{n-1} \quad \forall n$.

Then:

If $h_n \rightarrow 0$, then $\mathbf{E}\tau_+^W = \infty$.

If $\lim h_n \equiv c > 0$, then $\mathbf{E}\tau_+^W < \infty$.

Remark 2 Similar observations — for $\tau_+^S, \tau_-^W, \tau_-^S$.

Remark 3 Set $c = \mathbf{P}(\tau_+^S > \tau_+^W) \equiv \mathbf{P}(u_+^W = 0)$. Then $c < 1$.

Indeed, $\mathbf{P}(u_+^W > 0) \geq \mathbf{P}(\xi_1 > 0) > 0$.

Problem No 1. Set $c_- = \mathbf{P}(\tau_-^S > \tau_-^W)$. Prove: $c = c_-$!

Lemma 3 $\mathbf{P}(\tau_+^W < \infty) = 1 \iff \mathbf{P}(\tau_+^S < \infty) = 1$.

Proof. $\Leftarrow$. $\tau_+^S \geq \tau_+^W$ a.s.

$\Rightarrow$.

$$\begin{aligned} \tau_+^S &= \mu^{(1)} \cdot I(u^{(1)} > 0) + (\mu^{(1)} + \mu^{(2)}) \cdot I(u^{(1)} = 0, u^{(2)} > 0) + \\ &\dots + (\mu^{(1)} + \dots + \mu^{(m)}) \cdot I(u^{(1)} = 0, \dots, u^{(m-1)} = 0, u^{(2)} > 0) + \dots \\ &\equiv \mu^{(1)} + \mu^{(2)} \cdot I(u^{(1)} = 0) + \mu^{(3)} \cdot I(u^{(1)} = 0, u^{(2)} = 0) + \dots \end{aligned}$$

$$(*) \quad \tau_+^S = \sum_{i=1}^{\infty} \mu^{(i)} \cdot \prod_{j=1}^{i-1} I(u^{(j)} = 0) \quad \text{and} \quad \forall m = 1, 2, \dots,$$

$$\mathbf{P}(\tau_+^S < \infty) \geq \mathbf{P}(\tau_+^S \leq \sum_{i=1}^m \mu^{(i)}) = 1 - \mathbf{P}\left(\bigcap_{j=1}^{m-1} \{u^{(j)} = 0\}\right) = 1 - c^{m-1} \rightarrow 1$$

□

Corollary 2 Assume $\mathbf{P}(\tau_+^W < \infty) = 1$.

Then

$$\mathbf{P}(u_+^S > x) = \frac{1}{1-c} \cdot \mathbf{P}(u_+^W > x) \quad \forall x \geq 0.$$

Indeed,

$$\begin{aligned} \mathbf{P}(u_+^S > x) &= \mathbf{P}(u^{(1)} > x) + \mathbf{P}(u^{(1)} = 0, u^{(2)} > x) + \dots \\ &= \mathbf{P}(u^{(1)} > x)(1 + c + c^2 + \dots) = \frac{1}{1-c} \cdot \mathbf{P}(u_+^W > x). \end{aligned}$$

Therefore,

Corollary 3 $\forall r > 0, \quad \mathbf{E}\{(u_+^W)^r\} = \mathbf{E}\{(u_+^S)^r\} \cdot (1-c).$

Indeed, $u_+^W \stackrel{D}{=} u_+^S \cdot \alpha$, where $\alpha \perp u_+^S$; $\alpha = \begin{cases} 1, & 1-c \\ 0, & c \end{cases}$ (since $\mathbf{P}(u_+^S \alpha > x) = \mathbf{P}(u_+^S > x, \alpha = 1) = \mathbf{P}(u_+^S > x) \cdot \mathbf{P}(\alpha = 1)$).

Therefore,

$$\mathbf{E}\{(u_+^W)^r\} = \mathbf{E}\{(u_+^S \cdot \alpha)^r\} = \mathbf{E}\{(u_+^S)^r \cdot \alpha\} = \mathbf{E}\{(u_+^S)^r\} \cdot \mathbf{E}\alpha.$$

□

Corollary 4 From (*),

$$\mathbf{E}\tau_+^S = \sum_{i=1}^{\infty} \mathbf{E} \left\{ \mu^{(i)} \cdot \prod_{j=1}^{i-1} I(u^{(j)} = 0) \right\} = \mathbf{E}\mu^{(1)}(1+c+\dots+c^m+\dots) = \frac{1}{1-c} \mathbf{E}\tau_+^W.$$

Lemma 4 $\mathbf{P}(\tau_-^S < \infty) < 1 \iff \mathbf{P}(\tau_+^S < \infty) = 1 \text{ and } \mathbf{E}\tau_+^S < \infty.$

Corollary 5 There exist 3 possibilities only:

$$\begin{array}{lll} \text{("+") } & \mathbf{P}(\tau_+^S < \infty) = 1; & \text{("−") } \mathbf{P}(\tau_+^S < \infty) < 1; & \text{("0") } \mathbf{P}(\tau_+^S < \infty) = 1 \\ & \mathbf{E}\tau_+^S < \infty; & \mathbf{P}(\tau_-^S < \infty) = 1; & \mathbf{E}\tau_+^S = \infty; \\ & \mathbf{P}(\tau_-^S < \infty) < 1; & \mathbf{E}\tau_-^S < \infty; & \mathbf{P}(\tau_-^S < \infty) = 1 \\ & & & \mathbf{E}\tau_-^S = \infty; \end{array}$$

Proof of Lemma 4. Assume $\mathbf{P}(\tau_-^S < \infty) < 1$. It means:

$$\mathbf{P}(S_1 \geq 0, S_2 \geq 0, \dots, S_n \geq 0, \dots) > 0.$$

But $h_n = \mathbf{P}(S_1 \geq 0, \dots, S_n \geq 0) \geq h > 0 \ \forall n$.

Then $\frac{H_\mu(n)}{n} \geq h \ \forall n \implies$ from Lemma 2 and its corollary, $\mathbf{E}\tau_+^W < \infty \implies$ from Lemma 3

and Corollary 4, $\mathbf{E}\tau_+^S < \infty$.

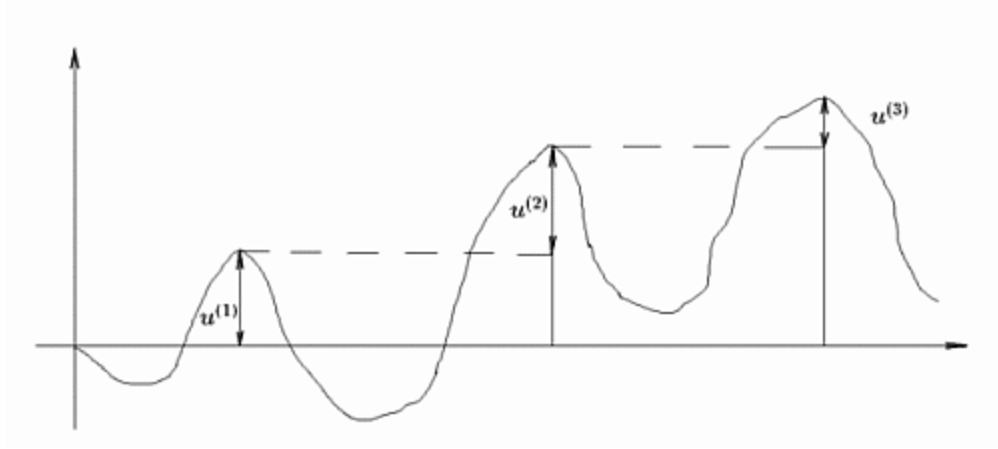
Vise versa...

□

Remark 4 (a) If $\mathbf{P}(\tau_+^S < \infty) = 1$, then $\overline{S} = \infty$ a.s.

(b) If $\mathbf{P}(\tau_+^S < \infty) < 1$, then $\overline{S} < \infty$ a.s.

and $\nu = \min\{n : S_n = \overline{S}\} < \infty$ a.s.



Indeed,

(a) $u^{(1)}, u^{(2)}, u^{(3)}$ are i.i.d., $\mathbf{P}(u^{(1)} > 0) = 1$,

$$\sum_{i=1}^m u^{(i)} \leq \overline{S} \ \forall m; \sum_{i=1}^m u^{(i)} \rightarrow \infty \text{ a.s.}$$

(b) Let $\phi + 1 = \min\{n : \mu^{(n)} < \infty\}$, $\nu = \sum_1^\phi \mu^{(i)}$. Then

$$\overline{S} = S_\nu.$$

Assume $\mathbf{E}|\xi_1| < \infty$.

Lemma 5 *If $\mathbf{E}\xi_1 > 0$, then $\mathbf{P}(\tau_+^S < \infty) = 1$ and $\mathbf{E}\tau_+^S < \infty$.*

Proof.

(1). Without loss of generality, one can assume that $\exists N \gg 1 : \mathbf{P}(\xi_1 \leq N) = 1$.

Indeed, if it is not true, then choose $N \gg 1 : \mathbf{E}\{\min(\xi_1, N)\} > 0$ and introduce

$$\tilde{\xi}_n = \min(\xi_n, N), \quad \tilde{S}_n = \tilde{\xi}_1 + \dots + \tilde{\xi}_n, \quad \tilde{\tau}_+^S = \min\{n \geq 1 : \tilde{S}_n > 0\}.$$

Since $\tilde{\xi}_n \leq \xi_n$ a.s. $\forall n \implies \tilde{\tau}_+^S \geq \tau_+^S$ a.s.

(2). So, assume $\mathbf{P}(\xi_1 \leq N) = 1$ for some $N < \infty$. SLLN:

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mathbf{E}\xi_1 > 0 \implies \tau_+^S < \infty \text{ a.s.}$$

Note: since τ_+^S is positive and integer-valued, $\mathbf{E}\tau_+^S = \sum_{i=1}^{\infty} \mathbf{P}(\tau_+^S \geq i)$.

Fix an arbitrary $n = 1, 2, \dots$ and set

$$V_n = \sum_1^n \xi_i \cdot I(\tau_+^S \geq i) \equiv \sum_1^{\min(n, \tau_+^S)} \xi_i.$$

First,

$$|V_n| \leq \sum_1^n |\xi_i| \implies \mathbf{E}|V_n| < \infty.$$

Second,

$$V_n = u_+^S, \text{ if } n \geq \tau_+^S, \text{ and } V_n \leq 0 < u_+^S, \text{ if } n < \tau_+^S.$$

Third,

$$0 < u_+^S \leq N \text{ a.s.}$$

Therefore, $V_n \leq u_+^S$ a.s. and $\mathbf{E}V_n \leq \mathbf{E}u_+^S \leq N$. But

$$\mathbf{E}V_n = \sum_1^n \mathbf{E}(\xi_i \cdot I(\tau_+^S \geq i)) = \sum_1^n \mathbf{E}\xi_i \cdot \mathbf{P}(\tau_+^S \geq i) = \mathbf{E}\xi_1 \cdot \sum_1^n \mathbf{P}(\tau_+^S \geq i).$$

Then $\sum_1^n \mathbf{P}(\tau_+^S \geq i) \leq N/\mathbf{E}\xi_1 \equiv \text{const} < \infty \forall n$.

Then $\lim_{n \rightarrow \infty} \sum_1^n \mathbf{P}(\tau_+^S \geq i) = \mathbf{E}\tau_+^S < \infty$. $\square$

Lemma 6 *If $\mathbf{E}\xi_1 = 0$, then $\mathbf{P}(\tau_+^S < \infty) = \mathbf{P}(\tau_-^S < \infty) = 1$.*

Proof. Assume, by contradiction, that $\mathbf{P}(\tau_-^S < \infty) < 1$ (the case $\mathbf{P}(\tau_+^S < \infty) < 1$ is symmetric to that).

$\implies \mathbf{E}\tau_+^S < \infty \implies$ From Wald identity,

$$0 < \mathbf{E}u_+^S = \mathbf{E}\xi_1 \cdot \mathbf{E}\tau_+^S = 0$$

$\implies$ contradiction $\square$

Corollary 6 *A simple random walk.*

1.3 Random Walks. Existence of moments of hitting times

Let $\{\xi_n\}$ be i.i.d.,

$\xi_i = \xi_i^+ - \xi_i^-$, $\mathbf{E}\xi_1 = a < 0$, $\tau \equiv \tau_-^S = \min\{n \geq 1 : S_n < 0\}$, $u = S_\tau$.

Theorem 1 *Let $k \geq 1$. If $\mathbf{E}(\xi_1^+)^k < \infty$, then $\mathbf{E}\tau^k < \infty$.*

Proof.

1) For $k = 1$ — was proved. $\boxed{k > 1}$.

2) For $L \gg 1$, put $\xi'_n = \max(\xi_n, -L) \geq \xi_n$ a.s., $S'_n = \xi'_1 + \dots + \xi'_n \geq S_n$ a.s. $\forall n$ and $\tau' = \min\{n \geq 1 : S'_n < 0\} \geq \tau$ a.s.

Choose L such that $\mathbf{E}\xi'_n = a' < 0$.

If we prove the statement of the Theorem for τ' $\implies$ we prove it for τ !

3) So, without loss of generality, one can assume that

$$\exists L > 0 : \mathbf{P}(\xi_n \geq -L) = 1 \implies \mathbf{P}(0 < -u \leq L) = 1!$$

4) $a\tau \equiv a\tau - u + u$.

Make use of the following algebraic inequality:

$$|\alpha + \beta|^k \leq 2^{k-1}(|\alpha|^k + |\beta|^k)$$

and take $\alpha = a\tau - u$, $\beta = u$.

$$\implies |a|^k \tau^k \leq 2^{k-1}(|a\tau - u|^k + |u|^k) \leq 2^{k-1}(|a\tau - u|^k + L^k)$$

$$\implies |a|^k \mathbf{E}(\tau^k) \leq 2^{k-1}(\mathbf{E}(|a\tau - u|^k) + L^k)$$

$$\implies \text{It is sufficient to prove a finiteness of } \mathbf{E}(|a\tau - u|^k).$$

5) $\tau_n = \min(\tau, n)$. Since τ is a ST, τ_n is, too ($\forall n$).

$u - a\tau = S_\tau - a\tau = \sum_{i=1}^{\tau} (\xi_i - a)$. Put $\psi_i = \xi_i - a$.

Note: $\mathbf{E}\psi_i = 0$, $\mathbf{E}|\psi_i|^k \leq 2^{k-1}(\mathbf{E}|\xi_i|^k + |a|^k) < \infty$. **(?? see manual)**

Put

$$V_n = \sum_1^{\tau_n} \psi_i \equiv \sum_1^n \psi_i \cdot I(\tau \geq i) \equiv V_{n-1} + \psi_n \cdot I(\tau \geq n).$$

Note: $\mathbf{E}V_n = \mathbf{E}\tau_n \cdot \mathbf{E}\psi_1 = 0$ (since $0 \leq \tau_n \leq \tau$, $\mathbf{E}\tau < \infty$)

and

$$\mathbf{E}(|V_n|^k) \leq \mathbf{E}\left(\left(\sum_1^n |\psi_i|\right)^k\right) \leq \dots < \infty \quad \forall n.$$

6) Make use of the other algebraic inequality:

$$|\alpha + \beta|^k \leq |\alpha|^k + C_{1,k} \cdot |\alpha|^{k-1} \cdot \text{sign}(\alpha) \cdot \beta + C_{2,k} |\beta|^k,$$

where $C_{1,k}$ and $C_{2,k}$ are constants that depend on k only,

$$\text{sign}(\alpha) = \begin{cases} 1, & \text{if } \alpha > 0 \\ 0, & \text{if } \alpha = 0 \\ -1, & \text{if } \alpha < 0. \end{cases}$$

Take $\alpha = V_{n-1}$, $\beta = \psi_n \cdot I(\tau \geq n)$.

$$\begin{aligned} \mathbf{E}(|V_n|^k) &\leq \mathbf{E}(|V_{n-1}|^k) + C_{2,k} \cdot \mathbf{E}\left\{|\psi_n|^k \cdot I(\tau \geq n)\right\} + \\ &+ C_{1,k} \cdot \mathbf{E}\left\{|V_{n-1}|^{k-1} \cdot \text{sign}(V_{n-1}) \cdot I(\tau \geq n) \cdot \psi_n\right\}. \end{aligned}$$

Set $C = \max(C_{2,k}, 1)$.

Note: if $\mathbf{E}|\phi_1| < \infty$, $\mathbf{E}|\phi_2| < \infty$, $\phi_1 \perp \phi_2 \implies \mathbf{E}\phi_1\phi_2 = \mathbf{E}\phi_1 \cdot \mathbf{E}\phi_2$.

(a) Since ψ_n and $I(\tau \geq n)$ are independent $\implies |\psi_n|^k$ and $I(\tau \geq n)$ are

$$\begin{aligned} \implies C_{2,k} \cdot \mathbf{E} \left\{ |\psi_n|^k \cdot I(\tau \geq n) \right\} &= C_{2,k} \cdot \mathbf{E} \left\{ |\psi_n|^k \right\} \cdot \mathbf{P}(\tau \geq n) \leq \\ &\leq C \cdot \mathbf{E} \left\{ |\psi_n|^k \right\} \cdot \mathbf{P}(\tau \geq n) < \infty \end{aligned}$$

(b) Since $|V_{n-1}|^{k-1} \cdot \text{sign}(V_{n-1}) \cdot I(\tau \geq n)$ are independent,

$$\mathbf{E}(|\dots|) \leq \mathbf{E}(\max(|V_{n-1}|^{k-1}, 1)) \leq \mathbf{E}(\max(|V_{n-1}|^k, 1)) \leq \mathbf{E}|V_{n-1}|^k + 1 < \infty,$$

$$\implies \mathbf{E}\{\dots \cdot \psi_n\} = \mathbf{E}\{\dots\} \cdot \mathbf{E}\psi_n = 0.$$

Therefore,

$$\mathbf{E}(|V_n|^k) \leq \mathbf{E}(|V_{n-1}|^k) + C \cdot \mathbf{E}(|\psi_1|^k) \cdot \mathbf{P}(\tau \geq n)$$

$$\begin{aligned} \implies \text{induction... } \mathbf{E}(|V_n|^k) &\leq C \cdot \mathbf{E}(|\psi_1|^k) \cdot \sum_{i=1}^n \mathbf{P}(\tau \geq i) \\ &\leq K \equiv C \cdot \mathbf{E}(|\psi_1|^k) \cdot \mathbf{E}\tau < \infty \quad \forall n. \end{aligned}$$

7) Set

$$Z_n = |u - a\tau|^k \cdot I(\tau \leq n).$$

Note:

$$Z_n = \begin{cases} |V_n|^k, & \text{if } \tau \leq n \\ 0, & \text{otherwise} \end{cases} \implies Z_n \leq |V_n|^k \text{ a.s.}$$

$$\implies \mathbf{E}Z_n \leq K \quad \forall n.$$

Next Part