

1 Elements of renewal theory

1.1 Notations and definitions

Let $\{\xi_n\}_{n \geq 1}$ be an i.i.d. sequence of r.v.'s; $\mathbf{P}(\xi_n \geq 0) = 1$; $\mathbf{P}(\xi_n > 0) > 0$.

Let $\xi_0 \perp \{\xi_n\}_{n \geq 1}$; $\mathbf{P}(\xi_0 \geq 0) = 1$. Set $S_0 = \xi_0$; $S_n = \xi_0 + \xi_1 + \dots + \xi_n$, $n \geq 1$.

S_n is

undelayed $\iff \xi_0 = 0$ a.s. (or zero-delayed)

delayed $\iff \mathbf{P}(\xi_0 > 0) > 0$.

$$\eta(t) = I(S_0 \leq t) + \max\{n \geq 0 : S_n \leq t\} \equiv \#\{n \geq 0 : S_n \leq t\} \equiv \sum_{k=0}^{\infty} I(S_k \leq t)$$

is a renewal process, $t \geq 0$.

$\{\eta(t), t \geq 0\}$ is (un)delayed $\iff \{S_n\}$ is (un)delayed.

$$H(t) = \mathbf{E}\eta(t) \equiv \mathbf{E} \left\{ \sum_{k=0}^{\infty} I(S_k \leq t) \right\} = \sum_{k=0}^{\infty} \mathbf{E} \{I(S_k \leq t)\} = \sum_{k=0}^{\infty} \mathbf{P}(S_k \leq t)$$

is a renewal function.

Both $\eta(t)$ and $H(t)$ are monotone $\nearrow$, right-continuous.

Examples.

[1] Deterministic undelayed renewal process:

$$\xi_0 \equiv 0; \xi_n \equiv 1, n \geq 1 \implies \eta(t) = [t] + 1 \equiv H(t).$$

[2] Homogeneous Poisson process:

$$\xi_0 \equiv 0; \xi_n \in \text{Exp}(\lambda), n \geq 1 \implies \eta(t) - 1 \in \Pi(\lambda t); H(t) = \lambda t + 1.$$

Show that any renewal process is well-defined and has “good properties”.

Remark 1 For $t < 0$, put $\eta(t) \equiv 0$ and $H(t) \equiv 0$.

Remark 2 By definition, $1 + \max\{n \geq 0 : S_n \leq t\} = \min\{n \geq 0 : S_n > t\}$.

Theorem 1 (i) $\forall$ renewal process, $\forall t > 0$,

$$\mathbf{P}(\eta(t) < \infty) = 1 \quad \text{and} \quad H(t) < \infty.$$

(ii) Moreover, $\exists c > 0$:

$$\mathbf{E}(\exp\{c\eta(t)\}) < \infty \quad \forall t > 0.$$

In particular,

$$\mathbf{E}(\eta(t)^k) < \infty \quad \forall k > 0, \forall t.$$

Proof. Without loss of generality, one can consider undelayed renewal processes only. Indeed,...

Let $\{S_n\}$ be an undelayed sequence and $\{\eta(t), t \geq 0\}$ — corresponding undelayed process.

1). $\mathbf{P}(\xi_n > 0) > 0 \implies \exists \varepsilon > 0 : \mathbf{P}(\xi_n \geq \varepsilon) \equiv p > 0$. Introduce r.v.'s ξ_n^* :

$$\xi_n^* = \begin{cases} \varepsilon, & \text{if } \xi_n \geq \varepsilon \\ 0, & \text{if } 0 \leq \xi_n < \varepsilon \end{cases} \equiv \begin{cases} \varepsilon, & \text{w.pr. } p > 0 \\ 0, & \text{w.pr. } q \equiv 1 - p. \end{cases}$$

$$\xi_n^* \leq \xi_n \text{ a.s. } \forall n \implies S_n^* \equiv \xi_1^* + \dots + \xi_n^* \leq S_n \text{ a.s.}$$

$$\implies \eta^*(t) = \#\{n : S_n^* \leq t\} \geq \#\{n : S_n \leq t\} = \eta(t) \text{ a.s. } \forall t$$

$\implies$ it is sufficient to prove the theorem for $\eta^*(t)$.

2). Put

$$\hat{\xi}_n = \frac{\xi_n^*}{\varepsilon} = \begin{cases} 1, & \text{w.pr. } p > 0 \\ 0, & \text{w.pr. } q \equiv 1 - p \end{cases}; \quad \hat{S}_n \equiv \hat{\xi}_1 + \dots + \hat{\xi}_n;$$

$$\hat{\eta}(t) = \#\{n : \hat{S}_n \leq t\} \equiv \#\{n : \frac{S_n^*}{\varepsilon} \leq t\} \equiv \#\{n : S_n^* \leq \varepsilon t\} = \eta^*(\varepsilon t)$$

$\implies$ it is sufficient to prove the theorem for $\hat{\eta}(t)$.

3) $\hat{\eta}(t) = \hat{\eta}([t]) \forall t \implies$ one can consider integer values of t only.

4) Set

$$\nu_1 = \min\{n : \hat{S}_n = 1\}, \quad \nu_2 = \min\{n : \hat{S}_n = 2\} - \nu_1, \dots,$$

$$\nu_{m+1} = \min\{n : \hat{S}_n = m + 1\} - \nu_1 - \nu_2 - \dots - \nu_m, \quad m \geq 2.$$

Then

(i) $\nu_1, \nu_2, \dots, \nu_m, \dots$ are i.i.d. r.v.'s;

(ii) $\nu_i \in G(p)$;

(iii) $\hat{\eta}(t) = \nu_1 + \dots + \nu_t + \nu_{t+1} < \infty$ a.s.

(iv) $\widehat{H}(t) = \mathbf{E}\hat{\eta}(t) \leq (t+1)\mathbf{E}\nu_1 = \frac{t+1}{p} < \infty$.

Moreover, for $c > 0, t > 0$,

$$\begin{aligned} \mathbf{E}\{\exp(c\hat{\eta}(t))\} &\leq \mathbf{E}\{\exp(c(\nu_1 + \dots + \nu_{t+1}))\} = \\ &= \mathbf{E}\left\{\prod_{i=1}^{t+1} \exp(c\nu_i)\right\} = \prod_{i=1}^{t+1} \mathbf{E}\{\exp(c\nu_i)\} = [\mathbf{E}\{\exp(c\nu_1)\}]^{t+1} \end{aligned}$$

and

$$\mathbf{E}\{\exp(c\nu_1)\} = \sum_{k=1}^{\infty} \exp(ck) \cdot p \cdot q^{k-1} = \begin{cases} pe^c, & \text{if } q = 0 \\ \frac{p}{q} \sum_{k=1}^{\infty} (e^c \cdot q)^k, & \text{if } q > 0. \end{cases}$$

Since $q < 1$

$$\implies \text{take } c < -\log q.$$

□

Definition 1 A r.v. ξ has a lattice (arithmetic) distribution, if $\exists h > 0$:

$$\sum_{k=-\infty}^{\infty} \mathbf{P}(\xi = kh) = 1. \quad (1)$$

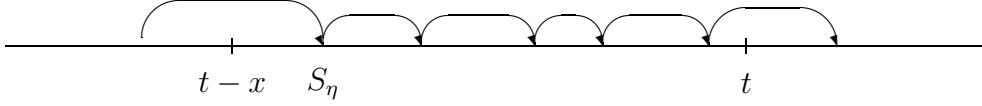
... has a lattice distribution with span h_0 , if h_0 is minimal between all numbers h which satisfy (1).

... has a non-lattice distribution, if ...

Lemma 1 For any undelayed renewal process and for any $0 \leq x \leq t$,

$$H(t) - H(t - x) \leq H(x) \quad \text{and} \quad \eta(t) - \eta(t - x) \leq_{\text{st}} \eta(x).$$

Proof. Let $\eta = \eta(t - x) = \min\{n : S_n > t - x\}$.



$$\begin{aligned} H(t) - H(t - x) &= \mathbf{E} \{ \#\{n : t - x < S_n \leq t\} \} = \\ &= \mathbf{E} \{ \#\{n : t - x < S_n \leq t\} \cdot I(S_\eta \leq t) \} = \mathbf{E} \{ \#\{n \geq \eta : S_n \leq t\} \cdot I(S_\eta \leq t) \} = \\ &= \mathbf{E} \{ (1 + \#\{n > \eta : S_n \leq t\}) \cdot I(S_\eta \leq t) \} \leq \\ &\leq 1 + \mathbf{E} \{ \#\{n > \eta : S_n \leq t\} \cdot I(S_\eta \leq t) \} = \\ &= 1 + \mathbf{E} \{ \#\{n > \eta : S_n - S_\eta \leq t - S_\eta\} \cdot I(S_\eta \leq t) \} \leq \\ &\leq 1 + \mathbf{E} \{ \#\{n > \eta : S_n - S_\eta \leq x\} \cdot I(S_\eta \leq t) \}. \end{aligned}$$

Put $\tilde{\xi}_i = \xi_{\eta+i}, i \geq 1$,
 $\tilde{S}_i = \tilde{\xi}_1 + \dots + \tilde{\xi}_i, i \geq 1$.

Then $\{\xi_i\}_{i \geq 1}$ are independent of S_η and

$$\#\{n > \eta : S_n - S_\eta \leq x\} \cdot I(S_\eta \leq t) = \#\{i \geq 1 : \tilde{S}_i \leq x\} \cdot I(S_\eta \leq t).$$

Therefore,

$$\begin{aligned} H(t) - H(t - x) &\leq 1 + \mathbf{E} \{ \#\{i \geq 1 : \tilde{S}_i \leq x\} \cdot I(S_\eta \leq t) \} \\ &\leq 1 + \mathbf{E} \{ \#\{i \geq 1 : \tilde{S}_i \leq x\} \} = H(x). \end{aligned}$$

□

Consider undelayed renewal process.

Law of large numbers (LLN) for renewal process:

Theorem 2. (LLN for RP)

(i) If $\mathbf{E}\xi_1 \equiv a < \infty$, then

$$\frac{\eta(t)}{t} \xrightarrow{\text{P}} \frac{1}{a}.$$

(ii) If $\mathbf{E}\xi_1 = \infty$, then

$$\frac{\eta(t)}{t} \xrightarrow{\text{P}} 0.$$

Strong law of large numbers (SLLN) for renewal process:

Theorem 2'. (SLLN for RP)

(i) ... $\xrightarrow{\text{a.s.}}$... **Problem No 1**

(ii) ... $\xrightarrow{\text{a.s.}}$...

Central limit theorem (CLT) for renewal process:

Theorem 3. (CLT for RP) If $\mathbf{E}\xi_1 \equiv a < \infty$ and $\mathbf{D}\xi_1 \equiv \mathbf{E}(\xi_1 - a)^2 \equiv \sigma^2 < \infty$, then $\forall x > 0$

$$\mathbf{P} \left(\frac{\eta(t) - t/a}{\sqrt{\frac{t\sigma^2}{a^3}}} \leq x \right) \rightarrow \Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\{-y^2/2\} dy \text{ as } t \rightarrow \infty$$

(in slightly other terms, $\frac{\eta(t) - t/a}{\sqrt{\frac{t\sigma^2}{a^3}}} \Rightarrow \psi \in N(0, 1)$)

Integral renewal theorem:

Theorem 4. (IRT)

$$\frac{H(t)}{t} \rightarrow \frac{1}{a} \text{ as } t \rightarrow \infty; \quad a = \mathbf{E}\xi_1 \leq \infty.$$

Local renewal theorem:

Theorem 5. (LRT) or Blackwell's RT Assume $a < \infty$.

- (a) ξ_n have a lattice distribution with span $h > 0 \implies H((n+1)h) - H(nh) \rightarrow \frac{h}{a}$ as $n \rightarrow \infty$
- (b) ξ_n have a non-lattice distribution $\implies H(t+u) - H(t) \rightarrow \frac{u}{a} \forall u > 0$ as $t \rightarrow \infty$

Theorem 6 Assume $a < \infty$ and $b = \mathbf{E}\xi_1^2 < \infty$.

- (b) ξ_n have a non-lattice distribution $\implies H(t) - \frac{t}{a} \rightarrow \frac{b}{2a^2}$ as $t \rightarrow \infty$
- (a) ξ_n have a lattice distribution with span $h > 0 \implies H(nh) - \frac{nh}{a} \rightarrow \dots$ as $n \rightarrow \infty$

Comments... 1) What for delayed ...

2) How we will proceed with proofs.

Proofs of Th's 2–4 $\longrightarrow$ Definitions of stationary RP $\longrightarrow$ Coupling of two renewal processes $\longrightarrow$ Proofs of Th. 5(a) and of Th. 6

The proof of Th. 5(b) will be done later on.

Proof of Theorem 2.

$$(i). \text{ LLN: } \frac{S_n}{n} \xrightarrow{p} a, \quad \text{SLLN: } \frac{S_n}{n} \xrightarrow{\text{a.s.}} a.$$

The following two events coincide:

$$\{\eta(t) > k\} = \{S_k \leq t\}.$$

Therefore, $\forall \varepsilon > 0$

$$\mathbf{P} \left(\left| \frac{\eta(t)}{t} - \frac{1}{a} \right| > \varepsilon \right) = \mathbf{P} \left(\frac{\eta(t)}{t} - \frac{1}{a} > \varepsilon \right) + \mathbf{P} \left(\frac{\eta(t)}{t} - \frac{1}{a} < -\varepsilon \right) \equiv P_1(t) + P_2(t).$$

$$P_1(t) = \mathbf{P} \left(\frac{\eta(t)}{t} > \varepsilon + \frac{1}{a} \right) = \mathbf{P} \left(\eta(t) > t \left(\varepsilon + \frac{1}{a} \right) \right) =$$

$$= \mathbf{P} \left(\eta(t) > \left[t \left(\varepsilon + \frac{1}{a} \right) \right] \right).$$

$$\text{Put } l \equiv l(t) = \left[t \left(\varepsilon + \frac{1}{a} \right) \right].$$

$$P_1(t) = \mathbf{P}(\eta(t) > l) = \mathbf{P}(S_l \leq t) = \mathbf{P} \left(\frac{S_l}{l} \leq \frac{t}{l} \right).$$

Note:

$$\frac{t}{l} = \frac{t}{\left[t \left(\varepsilon + \frac{1}{a} \right) \right]} \leq \frac{t}{t \left(\varepsilon + \frac{1}{a} \right) - 1} = \frac{1}{\varepsilon + \frac{1}{a} - \frac{1}{t}}$$

and, for $t \geq \frac{2}{\varepsilon}$,

$$\frac{t}{l} \leq \frac{1}{\frac{\varepsilon}{2} + \frac{1}{a}} \equiv a - \left(a - \frac{1}{\frac{\varepsilon}{2} + \frac{1}{a}} \right) = a - \left(\frac{a\varepsilon}{\varepsilon + \frac{2}{a}} \right) \equiv a - \delta_1,$$

where $\delta_1 > 0$. So, for $t \geq \frac{2}{\varepsilon}$, $t \rightarrow \infty$, $l = l(t) \rightarrow \infty$

$$\mathbf{P} \left(\frac{S_l}{l} \leq \frac{t}{l} \right) \leq \mathbf{P} \left(\frac{S_l}{l} \leq a - \delta_1 \right) \rightarrow 0 \text{ and } P_1(t) \rightarrow 0.$$

Similar arguments show that $P_2(t) \rightarrow 0$. **Problem No 2:** Prove it!

(ii). Assume $\mathbf{E}\xi_1 = a = \infty$. We have to show that

$$\forall \varepsilon > 0, \quad \mathbf{P} \left(\frac{\eta(t)}{t} > \varepsilon \right) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Note:

$$\begin{aligned} \mathbf{E}\xi_1 &= \lim_{N \rightarrow \infty} \mathbf{E} \{ \xi_1 \cdot I(\xi_1 \leq N) \} \\ \Rightarrow \quad \exists N : a_N &\equiv \mathbf{E} \{ \xi_1 \cdot I(\xi_1 \leq N) \} \geq \frac{2}{\varepsilon} \\ \Rightarrow \quad \frac{1}{a_N} &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Put $\xi_n^* = \xi_n \cdot I(\xi_n \leq N)$, $\mathbf{E}\xi_n^* = a_N < \infty$,

$$S_n^* = \xi_1^* + \dots + \xi_n^* \quad \Longrightarrow \quad \eta^*(t) \geq \eta(t) \text{ a.s.}$$

and

$$\begin{aligned} \mathbf{P}\left(\frac{\eta(t)}{t} > \varepsilon\right) &\leq \mathbf{P}\left(\frac{\eta^*(t)}{t} > \varepsilon\right) \equiv \mathbf{P}\left(\frac{\eta^*(t)}{t} > \frac{1}{a_N} + \left(\varepsilon - \frac{1}{a_N}\right)\right) \leq \\ &\leq \mathbf{P}\left(\frac{\eta^*(t)}{t} > \frac{1}{a_N} + \frac{\varepsilon}{2}\right) \rightarrow 0 \quad (\text{as follows from (i)}). \end{aligned}$$

□

Proof of Theorem 3. **CLT** : $\frac{S_n - na}{\sqrt{n\sigma^2}} \Rightarrow \phi \in N(0, 1)$ as $n \rightarrow \infty$.

Now:

$$\mathbf{P}(\eta(t) > k) = \mathbf{P}(S_k \leq t) = \mathbf{P}\left(\frac{S_k - ak}{\sigma\sqrt{k}} \leq \frac{t - ak}{\sigma\sqrt{k}}\right).$$

Therefore,

$$\mathbf{P}\left(\frac{\eta(t) - t/a}{\sqrt{\frac{t\sigma^2}{a^3}}} > x\right) = \mathbf{P}\left(\eta(t) > \frac{t}{a} + x\sqrt{\frac{t\sigma^2}{a^3}}\right) \equiv P(t).$$

Set

$$l = l(t) = \left\lceil \frac{t}{a} + x\sqrt{\frac{t\sigma^2}{a^3}} \right\rceil.$$

$$P(t) = \mathbf{P}(\eta(t) > l) = \mathbf{P}(S_l \leq t) = \mathbf{P}\left(\frac{S_l - al}{\sigma\sqrt{l}} \leq \frac{t - al}{\sigma\sqrt{l}}\right).$$

Here

$$\frac{t - al}{\sigma\sqrt{l}} = \frac{t - a\left(\frac{t}{a} + x\sqrt{\frac{t\sigma^2}{a^3}}\right) + O(1)}{\sigma\sqrt{\frac{t}{a} + o(1)}} = \frac{-x\sigma\sqrt{\frac{t}{a}} + O(1)}{\sigma\sqrt{\frac{t}{a} + o(\sqrt{t})}} =$$

$$= -x + o(1).$$

Therefore,

$$P(t) = \mathbf{P} \left(\frac{S_l - al}{\sigma\sqrt{l}} \leq -x + o(1) \right) \rightarrow \Phi(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} \exp\{-y^2/2\} dy$$

and $1 - P(t) \rightarrow 1 - \Phi(-x) = \Phi(x)$. $\square$

Proof of Theorem 4. Let $\varepsilon > 0$ be defined as in the proof of Theorem 1. In order to simplifize some notations, assume $\varepsilon = 1$. Then $\xi_n^* = \hat{\xi}_n$.

(a). Since

$$\frac{\eta(t)}{t} \leq \frac{\eta([t] + 1)}{[t]} \equiv \frac{\eta([t] + 1)}{[t] + 1} \cdot \frac{[t] + 1}{[t]}$$

and

$$\frac{\eta(t)}{t} \geq \frac{\eta([t])}{[t] + 1} \equiv \frac{\eta([t])}{[t]} \cdot \frac{[t]}{[t] + 1},$$

then similar inequalities are valid for $H(t)$, too (replace η by H ewery-where) and, therefore, it is sufficient to prove Theorem 4 for t – integers only.

$\implies$ Instead of “ t ”, I’ll write “ n ”.

(b). We know:

$$\frac{\eta(t)}{t} \xrightarrow{p} \frac{1}{a}.$$

If we prove that they are UI $\implies \mathbf{E} \frac{\eta(n)}{n} = \frac{H(n)}{n} \rightarrow \frac{1}{a}$.

Let’s prove it:

We know (see the proof of Theorem 1 for notations):

$$0 \leq \frac{\eta(n)}{n} \leq \frac{\hat{\eta}(n)}{n} \leq \frac{\nu_1 + \dots + \nu_n + \nu_{n+1}}{n} \equiv$$

$$\equiv \frac{\nu_1 + \dots + \nu_n + \nu_{n+1}}{n+1} \cdot \frac{n+1}{n} \xrightarrow{p} \mathbf{E}\nu_1 = \frac{1}{p} \quad (\text{LLN})$$

and

$$\mathbf{E} \frac{\nu_1 + \dots + \nu_{n+1}}{n} = \frac{n+1}{n} \mathbf{E}\nu_1 \xrightarrow{p} \frac{1}{p}$$

$$\implies \left\{ \frac{\nu_1 + \dots + \nu_{n+1}}{n} \right\} \text{ are UI} \quad \implies \left\{ \frac{\eta(n)}{n} \right\} \text{ are UI}.$$

□

1.2 Stationary Renewal Processes

For a renewal process $\{\eta(t), t \geq 0\}$, for any $t \geq 0$ define the forward recurrence time (excess)

$$\chi(t) = S_{\eta(t)} - t, \quad \chi(t) > 0 \text{ a.s.}$$

and the backward recurrence time (defect)

$$\gamma(t) = t - S_{\eta(t)-1}, \quad \gamma(t) \geq 0 \text{ a.s.}$$

Definition 2 *We say that $\{\eta(t), t \geq 0\}$ is stationary (in continuous time), if $\{\chi(t), t \geq 0\}$ is stationary (in continuous time)*

Remark 3 *If $\{\chi(t), t \geq 0\}$ is stationary, then $\forall t \geq 0, x \geq 0$*

$$\eta(t+x) - \eta(t) \stackrel{D}{=} \eta(x) - \eta(0).$$

Indeed, $\forall t$, put

$$\xi_0^{(t)} = \chi(t) \quad \text{and} \quad \xi_n^{(t)} = \xi_{\eta(t)+n}, \quad n = 1, 2, \dots$$

Then (as we know),

$$\{\xi_n^{(t)}\}_{n \geq 1} \text{ are i.i.d. , } \quad \xi_n^{(t)} \stackrel{D}{=} \xi_1, \xi_0^{(t)} \perp \{\xi_n^{(t)}\}_{n \geq 1}.$$

$\implies$ If $\xi_0^{(t)} \stackrel{D}{=} \xi_0^{(0)}$, then $\{\xi_0^{(t)}, \{\xi_n^{(t)}\}_{n \geq 1}\}$ have the same distribution $\forall t$.

(More “correct” definition: “ $\{\eta(t), t \geq 0\}$ has stationary increments”).

Definition 3 Assume that ξ_1 has a lattice distribution with span $h > 0$, and $\mathbf{P}(\xi_0 \in \{h, 2h, 3h, \dots\}) = 1$. We say that $\{\eta(t), t = 0, h, 2h, \dots\}$ is stationary in discrete time, if $\{\chi(t), t = 0, h, 2h, \dots\}$ is stationary in discrete time.

Start with stationary RP in discrete time (DTSRP).

Lemma 2 If $\mathbf{P}(\xi_0 = mh) = \frac{h \cdot \mathbf{P}(\xi_1 \geq mh)}{a}$, then $\{\eta(t), t = 0, h, 2h, \dots\}$ is DTSRP.

Proof. Put $P_m = \mathbf{P}(\xi_1 \geq mh)$, $p_m = \mathbf{P}(\xi_1 = mh)$.

Induction arguments: it is sufficient to show that $\chi(1) \stackrel{D}{=} \xi_0$.

$$\begin{aligned} \mathbf{P}(\chi(1) \geq mh) &= \mathbf{P}(\xi_1 \geq (m+1)h) + \mathbf{P}(\xi_0 = h, \xi_1 \geq mh) + \mathbf{P}(\xi_0 = h, \xi_1 = 0, \xi_2 \geq mh) + \dots \\ &= \mathbf{P}(\xi_1 \geq (m+1)h) + \frac{P_1}{a}(P_m + p_0 P_m + p_0^2 P_m + \dots) = \\ &= \mathbf{P}(\xi_1 \geq (m+1)h) + \frac{P_m}{a} \\ &\quad (\text{Since } P_1 > 0, p_0 + P_1 = 1 \implies 1 + p_0 + p_0^2 + \dots = \frac{1}{P_1}). \end{aligned}$$

□

Remark 4 In the proof—“iff”.

Remark 5 For $h = 1$. (It is a scaling parametr, we can always ...

$$\dots \tilde{\xi}_n = \frac{\xi_n}{h}, \quad n \geq 0; \quad \mathbf{E}\tilde{\xi}_n = \frac{a}{h}, \quad n \geq 1)$$

Remark 6 For $h = 1$,

$$\begin{aligned} \mathbf{E}\xi_0 &= \sum_{m=1}^{\infty} m \cdot \mathbf{P}(\xi_0 = m) = \frac{1}{a} \sum_{m=1}^{\infty} m \cdot P_m = \frac{1}{a} \sum_{m=1}^{\infty} m \cdot \sum_{l=m}^{\infty} p_l = \\ &= \frac{1}{a} \sum_{l=1}^{\infty} p_l \cdot \sum_{m=1}^l m = \frac{1}{a} \sum_{l=1}^{\infty} p_l \cdot \frac{l^2 + l}{2} = \frac{1}{2a}(b + a), \end{aligned}$$

where $b = \mathbf{E}\xi_1^2$.

CTSRP (continuous time ...)

Lemma 3 Let $\{\xi_n\}_{n \geq 1}$ have an arbitrary distribution with finite mean $a = \mathbf{E}\xi_1$.

If

$$\mathbf{P}(\xi_0 > x) = \frac{1}{a} \int_x^\infty \mathbf{P}(\xi_1 > y) dy \quad \forall x \geq 0, \quad (2)$$

then $\{\eta(t), t \geq 0\}$ is CTSRP.

Remark 7 (a) ξ_0 has an absolutely continuous distribution with density

$$\frac{\mathbf{P}(\xi_1 > x)}{a} = \frac{1}{a}(1 - F(x)) \implies \text{in (2), } \mathbf{P}(\xi_0 > x) = \mathbf{P}(\xi_0 \geq x).$$

(b) $\mathbf{P}(\xi_1 > y)$ may be replaced by $\mathbf{P}(\xi_1 \geq y)$.

Proof.

1). For any $m = 1, 2, \dots$, introduce $\{\xi_n^{(m)}\}$:
take $h_m = \frac{1}{2^m}$ and put $\xi_n^{(m)} = lh_m$, if $\xi_n \in [lh_m, (l+1)h_m]$.

Note:

$$\xi_n^{(m)} \nearrow \xi_n \text{ a.s.} \quad \forall n \text{ as } m \rightarrow \infty.$$

2). $\forall m$, introduce $\xi_0^{(m)}$:

$$\mathbf{P}(\xi_0^{(m)} > kh_m) = \frac{h_m}{a} \cdot \mathbf{P}(\xi_1^{(m)} > kh_m).$$

Show that $\xi_0^{(m)} \Rightarrow \xi_0$ from (2).

We know: if $h_m \cdot k_m \rightarrow t$, $h_m \rightarrow 0$ and $f \geq 0$ is bounded and integrable,
then $\sum_{l=k_m}^\infty f(lh_m)h_m \rightarrow \int_t^\infty f(u)du$.

$$\text{Put } k_m = \left\lfloor \frac{t}{h_m} \right\rfloor + 1.$$

$$\mathbf{P}(\xi_0^{(m)} > t) = \mathbf{P}\left(\xi_0^{(m)} > \left\lfloor \frac{t}{h_m} \right\rfloor \cdot h_m\right) = \frac{1}{a^{(m)}} \sum_{l=k_m}^\infty \mathbf{P}(\xi_1^{(m)} > lh_m) \cdot h_m =$$

$$= \frac{1}{a^{(m)}} \sum_{l=k_m}^{\infty} \mathbf{P}(\xi_1 > lh_m) \cdot h_m \rightarrow \frac{1}{a} \int_t^{\infty} \mathbf{P}(\xi_1 > u) du.$$

3). Construct a coupling: $\xi_0^{(m)} \xrightarrow{\text{a.s.}} \xi_0$.

4). Since $S_0 = \xi_0$ has a continuous distribution $\implies$ the same for any $S_n, n \geq 1$.

Take any $t \geq 0$

$$\implies \mathbf{P}(\underbrace{\exists n : S_n = t}_A) \leq \sum \mathbf{P}(S_n = t) = 0$$

$$\implies \forall \omega \in \Omega \setminus A, \forall n = 0, 1, \dots, \eta(t)(\omega) < \infty,$$

$$S_n^{(m)} \rightarrow S_n \implies \eta^{(m)}(t) \rightarrow \eta(t) \text{ and } S_{\eta^{(m)}(t)} - t \rightarrow S_{\eta(t)} - t. \quad \square$$

Remark 8 If $\{\eta(nh), n = 0, 1, \dots\}$ is stationary in discrete time with span h , then

$$H(nh) = \frac{nh}{a} \quad \forall n = 0, 1, \dots$$

Indeed, $H(0) = 0$ and $H(nh) = H(h) - H(0) + H(2h) - H(h) + \dots + H(nh) - H((n-1)h) = n \cdot H(h)$. From Theorem 4,

$$\frac{H(nh)}{nh} \rightarrow \frac{1}{a} \implies H(h) = \frac{h}{a}.$$

Remark 9 If $\{\eta(t), t \geq 0\}$ is stationary in continuous time, then

$$H(t) = \frac{t}{a} \quad \forall t \geq 0.$$

Indeed, $H(0) = 0$ and, for $c = H(1)$, $H(n) = nc$; $H(1/m) = c/m$:

$$\left(H(1) = H\left(\frac{1}{m}\right) + H\left(\frac{2}{m}\right) - H\left(\frac{1}{m}\right) + \dots + H(1) - H\left(\frac{m-1}{m}\right) \right)$$

$$\implies H\left(\frac{n}{m}\right) = \frac{n}{m}c \text{ for any positive integer } n, m$$

$$\implies \text{since } H \text{ is right-continuous, } H(t) = tc \quad \forall t.$$

Proof of Theorem 5(a).

Assume that ξ_1 has a lattice distribution with span $h = 1$. It means:

(a) $\mathbf{P}(\xi_1 \in \{0, 1, 2, \dots\}) = 1$,

(b) G.C.D. $\{m : \mathbf{P}(\xi_1 = m) > 0\} = 1$. **Problem No 3:** Why?

Lemma 4 Let $\{\psi_n\}_{n=1}^\infty$ be an i.i.d. sequence of symmetric r.v.'s;

$$\psi_1 = \begin{cases} 1, & \text{w.pr. } \alpha \\ -1, & \text{w.pr. } \alpha \\ 0, & \text{w.pr. } r \end{cases} \quad 2\alpha + r = 1, \alpha > 0.$$

Consider a sequence $X_0 = k \in \mathbf{Z}$, $X_n = k + \sum_{i=1}^n \psi_i$, $n \geq 1$.

Then $\mu = \min\{n \geq 1 : X_n = 0\} < \infty$ a.s.

Proof — later on (in the section “Random Walks”).

1st step. Assume that $\exists i \geq 0 : p_i := \mathbf{P}(\xi_1 = i) > 0$ and $p_{i+1} > 0$.

Set

$$\alpha = \sum_{i=0}^{\infty} \alpha_i; \quad \alpha_i = \min(p_i, p_{i+1}) \quad \Longrightarrow \quad \alpha > 0.$$

We shall use the coupling from Lemma 9, Section “Stopping Times”.

Consider a sequence $\{\xi_{n,1}\}_{n \geq 1}$ and construct, on its base, $\{\xi_{n,2}\}_{n \geq 1}$:

$$\forall n \geq 1, \forall i \geq 0, \text{ if } \xi_{n,1} = i, \text{ then } \xi_{n,2} = \begin{cases} i-1, & \text{w.pr. } \alpha_{i-1}/p_i \\ i+1, & \text{w.pr. } \alpha_i/p_i \\ i, & \text{w.pr. } 1 - (\alpha_i + \alpha_{i-1})/p_i \end{cases}$$

(here $\alpha_{-1} = 0$).

In other words, we can introduce a family of r.v.'s $\{\beta_n(i)\}_{n \geq 1, i \geq 0}$ that are mutually independent; $\perp$ of $\{\xi_n\}$ and $\forall n, i$

$$\beta_n(i) = \begin{cases} -1, & \text{w.pr. } \alpha_{i-1}/p_i \quad (= 0, \text{ if } p_i = 0) \\ 1, & \text{w.pr. } \alpha_i/p_i \quad (= 0, \text{ if } p_i = 0) \\ i, & \text{w.pr. } 1 - (\alpha_i + \alpha_{i-1})/p_i \quad (= 1, \text{ if } p_i = 0) \end{cases}$$

Then — put $\xi_{n,2} = \xi_{n,1} + \beta_n(\xi_{n,1})$.

(i). Show that $\xi_{n,2} \stackrel{D}{=} \xi_{1,1}$.

Indeed,

$$\mathbf{P}(\xi_{n,2} = i) = \mathbf{P}(\xi_{n,2} = i, \xi_{n,1} = i+1) + \mathbf{P}(\xi_{n,2} = i, \xi_{n,1} = i)$$

$$\begin{aligned}
& + \mathbf{P}(\xi_{n,2} = i, \xi_{n,1} = i - 1) = \\
& = p_{i+1} \cdot \frac{\alpha_i}{p_{i+1}} + p_i \cdot \left(1 - \frac{\alpha_i + \alpha_{i+1}}{p_i}\right) + p_{i-1} \cdot \frac{\alpha_i}{p_{i-1}} = p_i.
\end{aligned}$$

(ii) $\mathbf{P}(\xi_{n,2} - \xi_{n,1} = \pm 1) = \alpha \quad \Longrightarrow \quad \text{Put } \psi_n = \xi_{n,2} - \xi_{n,1}.$

(iii) Let $m_1, m_2 \geq 0$ be two integers. Consider two renewal sequences

$$\xi_{0,1} = S_{0,1} = m_1, \quad S_{n,1} = m_1 + \sum_1^n \xi_{i,1}$$

$$\xi_{0,2} = S_{0,2} = m_2, \quad S_{n,2} = m_2 + \sum_1^n \xi_{i,2}$$

Then $X_n := S_{n,2} - S_{n,1} = k + \sum_1^n \psi_i$, where $k = m_1 - m_2 \Longrightarrow$ from Lemma above,

$$\mu = \min\{n \geq 1 : S_{n,1} = S_{n,2}\} = \min\{n \geq 1 : X_n = 0\} < \infty \text{ a.s.}$$

(iv) Assume, more generally, that $\xi_{0,1}$ and $\xi_{0,2}$ are r.v.'s. Then, for any coupling of $\xi_{0,1}, \xi_{0,2}$

$$\mathbf{P}(\nu < \infty) = \sum_{m_1, m_2} \mathbf{P}(\xi_{0,1} = m_1, \xi_{0,2} = m_2) \cdot \mathbf{P}(\mu < \infty \mid \xi_{0,1} = m_1, \xi_{0,2} = m_2) = 1.$$

(v) Take $\xi_{0,1} = 0, \xi_{0,2}$ — from the stationary version.

$$\text{Put } \xi_0 = \xi_{0,1} = 0, \xi_n = \begin{cases} \xi_{n,1}, & \text{if } n \leq \mu \\ \xi_{n,2}, & \text{if } n > \mu \end{cases}$$

Then

$$\begin{aligned}
& \mathbf{P}(\eta(k+l) - \eta(k) \in B) = \\
& = \mathbf{P}(\eta(k+l) - \eta(k) \in B; S_\mu \leq k) + \mathbf{P}(\eta(k+l) - \eta(k) \in B; S_\mu > k) \leq \\
& \leq \mathbf{P}(\eta_2(k+l) - \eta_2(k) \in B; S_\mu \leq k) + \mathbf{P}(S_\mu > k) \leq \\
& \leq \mathbf{P}(\eta_2(k+l) - \eta_2(k) \in B) + \mathbf{P}(S_\mu > k) = \\
& = \mathbf{P}(\eta_2(l) - \eta_2(0) \in B) + \mathbf{P}(S_\mu > k).
\end{aligned}$$

Similarly,

$$\mathbf{P}(\eta(k+l) - \eta(k) \in B) \geq \mathbf{P}(\eta(k+l) - \eta(k) \in B; S_\mu \leq k) =$$

$$\begin{aligned}
&= \mathbf{P}(\eta_2(k+l) - \eta_2(k) \in B) - \mathbf{P}(\eta_2(k+l) - \eta_2(k) \in B; S_\mu > k) \geq \\
&\geq \mathbf{P}(\eta_2(l) - \eta_2(0) \in B) - \mathbf{P}(S_\mu > k).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \sup_B |\mathbf{P}(\eta(k+l) - \eta(k) \in B) - \mathbf{P}(\eta_2(l) - \eta_2(0) \in B)| &\leq \mathbf{P}(S_\mu > k) \rightarrow 0 \\
(3)
\end{aligned}$$

as $k \rightarrow \infty$.

Finally, since $\{\eta(k+l) - \eta(k)\}_{k \geq 0}$ are UI,

$$H(k+l) - H(k) \rightarrow H_2(l) - H_2(0) = \frac{l}{a}.$$

□

Remark 10 Take $B = \{0\}$. Since $\{\eta(k+l) - \eta(k) = 0\} = \{\chi(k) > l\}$,

$$\Rightarrow \quad \mathbf{P}(\chi(k) \geq l) \rightarrow \mathbf{P}(\xi_{0,2} \geq l) = \frac{1}{a} \sum_{r=l}^{\infty} P_r.$$

Phenomenon: ...

Remark 11 Since $\{\gamma(k) \geq m, \chi(k) \geq l\} = \{\chi(k-m) \geq l+m\}$,

$$\begin{aligned}
\Rightarrow \quad &\mathbf{P}(\gamma(k) \geq m, \chi(k) \geq l) \rightarrow \frac{1}{a} \sum_{r=l+m}^{\infty} P_r \\
\Rightarrow \quad &\mathbf{P}(\gamma(k) \geq m, \chi(k) = l) = \\
&= \mathbf{P}(\gamma(k) \geq m, \chi(k) \geq l) - \mathbf{P}(\gamma(k) \geq m, \chi(k) \geq l+1) \\
&\quad \rightarrow \frac{1}{a} P_{m+l} = \frac{1}{a} \mathbf{P}(\xi_1 \geq m+l) \\
\Rightarrow \quad &\mathbf{P}(\gamma(k) = m, \chi(k) = l) = \\
&= \mathbf{P}(\gamma(k) \geq m, \chi(k) = l) - \mathbf{P}(\gamma(k) \geq m+1, \chi(k) = l) \\
&\quad \rightarrow \frac{1}{a} p_{m+l}
\end{aligned}$$

2nd step. What to do if $\alpha_i = 0 \ \forall i = 0, 1, \dots$?

Since $\text{G.C.D.}\{m \geq 1 : p_m > 0\} = 1$,

$\exists m_0 \geq 1 : \forall m \geq m_0, \exists l \equiv l(m); \exists i_1, \dots, i_l; \exists r_1, \dots, r_l \geq 0:$

$$p_{i_1} \cdot p_{i_2} \cdot \dots \cdot p_{i_l} > 0 \text{ and } m = r_1 \cdot i_1 + \dots + r_l \cdot i_l \\ \implies \mathbf{P}(S_{r_1+\dots+r_l} = m) \geq p_{i_1}^{r_1} \cdot \dots \cdot p_{i_l}^{r_l} > 0$$

So, we can take $m_1 = m_0$, $m_2 = m_0 + 1$ and consider “blocks” (see Lemma 10 from section “Stopping Times”).

Proof of Theorem 6 (due to T.Lindvall).

(b). Let ξ'_0 be independent of $\{\xi_n\}$ and $\xi'_n = \xi_n$, $n \geq 1$, $S'_n = \xi'_0 + \xi_1 + \dots + \xi_n$, $n \geq 1$; $\eta'(t) = \#\{n \geq 0 : S'_n \leq t\}$.

Assume that ξ'_0 has a density $\frac{1-F(x)}{a}$, $x \geq 0$. Then $\eta'(t)$ is stationary renewal process, and $H'(t) = \mathbf{E}\eta'(t) = t/a$, $t \geq 0$.

We have

$$\begin{aligned} H(t) - t/a &= H(t) - H'(t) = \mathbf{E}\eta(t) - \mathbf{E}\eta'(t) = \\ &= \mathbf{E}(\eta(t) - \eta'(t)) = \mathbf{E}\{\#\{n : t - \xi'_0 < S_n \leq t\}\}. \end{aligned}$$

Since

$$\begin{aligned} \mathbf{E}\{\#\{n : t - \xi'_0 < S_n \leq t\}\} &\equiv E(t) = \\ &= \mathbf{E}\left\{\mathbf{E}\left\{\#\{n : t - \xi'_0 < S_n \leq t\} \middle| \xi'_0\right\}\right\} = \\ &= \mathbf{E}\{H(t) - H(t - \xi'_0)\} \end{aligned}$$

and $H(t) - H(t - \xi'_0) \xrightarrow{\text{a.s.}} \frac{\xi'_0}{a}$ as $t \rightarrow \infty$ (see the Corollary of Theorem 5),

then $E(t) \rightarrow \mathbf{E}\frac{\xi'_0}{a} = \frac{b}{2a^2}$, if $\{H(t) - H(t - \xi'_0), t \geq 0\}$ are UI. Let's prove the latter fact.

(a). Note that $\mathbf{E}H(\xi'_0) < \infty$. Indeed,

$$\begin{aligned} \frac{H(t)}{t} &\rightarrow \frac{1}{a} \text{ as } t \rightarrow \infty \text{ and } H(t) \nearrow \\ \implies &\exists \text{ a constant } c : H(t) \leq c(1+t) \forall t \\ \implies &\mathbf{E}H(\xi'_0) \leq \mathbf{E}\{c(1+\xi'_0)\} = c(1+\mathbf{E}\xi'_0) < \text{infity}. \end{aligned}$$

(b). Note that $H(t) - H(t - x) \leq H(x) \forall x \geq 0$ (see Lemma 1). Therefore,

$$H(t) - H(t - \xi'_0) \leq H(0) + H(t) - H(\max(0, t - \xi'_0)) \leq H(t - \max(0, t - \xi'_0)) + H(0) \leq$$

$$\leq H(\xi'_0) + H(0) \quad \text{a.s.} \quad \text{for all } t.$$

$$\implies \{H(t) - H(t - \xi'_0), t \geq 0\} \text{ are UI.} \quad \square$$

Proof of Theorem 6(a) is similar to that of (b) (only one change: consider t on lattice only).

1.3 Transient renewal processes (terminating)

Assume that $\{\xi_n\}_{n \geq 1}$ are i.i.d. generalized r.v.'s (defective):

$$\mathbf{P}(\xi_n \geq 0) = 1, \quad \mathbf{P}(\xi_n < \infty) = p \leq 1, \quad \mathbf{P}(\xi_n = \infty) = q = 1 - p \geq 0.$$

$S_n = \xi_1 + \dots + \xi_n$ — undelayed sequence (smth. $+\infty = \infty, \infty + \infty = \infty$)
 $\eta(t)$ — renewal process, $H(t)$ — renewal function,

$$\nu : \{\nu = k\} = \{\xi_1 < \infty, \dots, \xi_k < \infty, \xi_{k+1} = \infty\}.$$

If $q > 0$, then

$$\mathbf{P}(\nu = k) = p^k q, \quad k \geq 0 \quad \text{and} \quad \mathbf{E}\nu = \frac{p}{q} < \infty.$$

Lemma 5 $\lim_{t \rightarrow \infty} H(t) < \infty \iff q > 0.$

Proof $\Leftarrow$.

$$q > 0 \implies \eta(\infty) = \nu \text{ and } \mathbf{E}\nu = \frac{p}{q} = \mathbf{E}\eta(\infty) \geq \mathbf{E}\eta(t) \quad \forall t.$$

$\Rightarrow$. Assume $q = 0$.

$$\implies \mathbf{P}(\eta(t) > k) = \mathbf{P}(S_k \leq t) \rightarrow 1 \text{ as } t \rightarrow \infty \quad \forall k.$$

$$\text{Set } \psi_k(t) = k \cdot I(\eta(t) > k) \equiv \begin{cases} k, & \text{if } \eta(t) > k \\ 0, & \text{otherwise.} \end{cases}$$

Then $\eta(t) \geq \psi_k(t)$ a.s. $\forall k$

$$\implies H(t) = \mathbf{E}\eta(t) \geq \mathbf{E}\psi_k(t) = k \cdot \mathbf{P}(\eta(t) > k) \rightarrow k \quad \forall k$$

$$\implies \liminf_{t \rightarrow \infty} H(t) \geq k \quad \forall k \implies \lim_{t \rightarrow \infty} H(t) = \infty.$$

□

Let $\{\xi_n\}$ be an i.i.d. sequence, $\mathbf{P}(\xi_n \geq 0) = 1, \mathbf{P}(\xi_n > 0) > 0$, and $S_n = \xi_1 + \dots + \xi_n, S_0 = 0$.

Let $\{\xi'_n\}$ be an independent copy of $\{\xi_n\}; S'_n = \xi'_1 + \dots + \xi'_n$.

Lemma 6 *If ξ_1 has a non-lattice distribution, then $\forall h > 0, \forall \delta \leq h \exists n_1, n_2 \geq 0$:*

$$\mathbf{P}(S'_{n_2} - S_{n_1} \in (h - \delta, h)) > 0.$$

APPENDIX (the Proof of Lemma)

Denote by $\mathcal{K}$ the support of $\{S_n\}_{n \geq 0}$, i.e.

$$\mathcal{K} = \{x \geq 0 : \forall \varepsilon > 0 \exists n : \mathbf{P}(S_n \in (x - \varepsilon, x + \varepsilon)) > 0\}.$$

Properties of $\mathcal{K}$:

1) $\mathcal{K}$ is unbounded.

Indeed, if $x \in \mathcal{K} \implies mx \in \mathcal{K} \quad \forall m = 2, 3, \dots$:

$$\begin{aligned} & \mathbf{P}(S_{mn} \in (mx - \varepsilon, mx + \varepsilon)) \geq \\ & \mathbf{P}\left(S_n \in \left(mx - \frac{\varepsilon}{m}, mx + \frac{\varepsilon}{m}\right), S_{2n} - S_n \in \left(mx - \frac{\varepsilon}{m}, mx + \frac{\varepsilon}{m}\right), \right. \\ & \quad \left. \dots, S_{mn} - S_{(m-1)n} \in (\dots)\right) = \\ & = \left[\mathbf{P}\left(S_n \in \left(mx - \frac{\varepsilon}{m}, mx + \frac{\varepsilon}{m}\right)\right) \right]^m > 0. \end{aligned}$$

2) $\mathcal{K}$ is closed set.

Assume $x_l \in \mathcal{K}, x_l \rightarrow x \implies \forall \varepsilon > 0, \exists l: |x_l - x| < \varepsilon/2$ and $\exists n : \mathbf{P}(S_n \in (x - \varepsilon/2, x + \varepsilon/2)) > 0$. But $(x - \varepsilon/2, x + \varepsilon/2) \subseteq (x - \varepsilon, x + \varepsilon)$

$$\implies \mathbf{P}(S_n \in (x - \varepsilon, x + \varepsilon)) > 0 \implies x \in \mathcal{K}.$$

- 3) If $x, y \in \mathcal{K}$, then for any positive integers m and l , $mx + ly \in \mathcal{K}$.
 Indeed, put $z = mx + ly$. Take any $\varepsilon > 0$, set $\varepsilon' = \frac{\varepsilon}{m+l}$ and choose n_1, n_2 :

$$\mathbf{P}(S_{n_1} \in (x - \varepsilon', x + \varepsilon')) \equiv p_1 > 0;$$

$$\mathbf{P}(S_{n_2} \in (y - \varepsilon', y + \varepsilon')) \equiv p_2 > 0.$$

Then

$$\begin{aligned} & \mathbf{P}(S_{mn_1+ln_2} \in (z - \varepsilon, z + \varepsilon)) \geq \\ & \geq \mathbf{P}(S_{n_1} \in (x - \varepsilon', x + \varepsilon'), S_{2n_1} - S_{n_1} \in (\dots), \dots, S_{mn_1} - S_{(m-1)n_1} \in (\dots); \\ & S_{mn_1+n_2} - S_{mn_1} \in (y - \varepsilon', y + \varepsilon'), \dots, S_{mn_1+ln_2} - S_{mn_1+(l-1)n_2} \in (\dots)) = \\ & = p_1^m p_2^l > 0. \end{aligned}$$

- 4) $0 \in \mathcal{K}$.

For a point x and a set A , define a distance d :

$$d(x, A) = \inf_{y \in A} |x - y|.$$

Theorem 7 *If ξ_1 has a non-lattice distribution, then the set $\mathcal{K}$ is asymptotically dense, i.e.*

$$d(x, \mathcal{K}) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Assume that the Theorem is proved. **Prove** Lemma.

$$\begin{aligned} & \forall \varepsilon > 0 \exists x_1 : d(x_1, \mathcal{K}) \leq \varepsilon \quad \forall x \geq x_1 \\ & \implies \exists y \in \mathcal{K} : d(x_1, \mathcal{K}) = |x_1 - y| \leq 2\varepsilon \\ & \implies \exists n_1 : \\ \mathbf{P}(A_1) & \equiv \mathbf{P}(S_{n_1} \in (x_1 - 3\varepsilon, x_1 + 3\varepsilon)) \geq \mathbf{P}(S_{n_1} \in (y - \varepsilon, y + \varepsilon)) > 0. \end{aligned}$$

Take $x_2 = x_1 + z$, $z > 0$ is arbitrary.

Similarly, $\exists n_2$:

$$\mathbf{P}(A_2) \equiv \mathbf{P}(S'_{n_2} \in (x_2 - 3\varepsilon, x_2 + 3\varepsilon)) > 0.$$

If $A_1 \cap A_2$ occurs, then

$$S'_{n_2} - S_{n_1} < x_2 + 3\varepsilon - (x_1 - 3\varepsilon) = z + 6\varepsilon$$

$$S'_{n_2} - S_{n_1} > x_2 - 3\varepsilon - (x_1 + 3\varepsilon) = z - 6\varepsilon$$

Therefore,

$$\mathbf{P}(S'_{n_2} - S_{n_1} \in (z - 6\varepsilon, z + 6\varepsilon)) \geq \mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1) \cdot \mathbf{P}(A_2) > 0.$$

$\Rightarrow$ for a given h, δ , take $\varepsilon = \delta/12$, $z = h - 6\varepsilon$. □

Now, let's prove the Theorem.

It is based on the following “arithmetical”

Lemma 7 $\forall$ closed unbounded set $A \subseteq \mathbf{R}_+$,

either

(a) $\exists$ a sequence $\{y_n\}$, $y_n \in A$ and $y \in A$:

$y_n \rightarrow y$ and $y_n \neq y \forall n$; $y_n \neq y_m \forall n \neq m$;

or

(b) $A = \{y_0, y_1, y_2, \dots\}$, where $y_n \rightarrow \infty$ as $n \rightarrow \infty$.

An elementary proof is omitted.

Lemma 8 If, for the set $\mathcal{K}$, (a) from Lemma 7 takes place, then $\mathcal{K}$ is asymptotically dense.

Proof. Without loss of generality, assume y_n to be strictly increasing, $y > 0$ (the proof of other cases is similar).

Put $\varepsilon_n = y - y_n$ and $A_{m,n} = [my_n, my)$ for $m = 1, 2, \dots$

Note that the points

$$my_n, (m-1)y_n + y, (m-2)y_n + 2y, \dots, y_n + (m-1)y, my$$

belong to $\mathcal{K}$ and divide $A_{m,n}$ into m intervals of length ε_n . So, for any m , if $x \in A_{m,n}$, then

$$d(x, \mathcal{K}) \leq \varepsilon_n.$$

Put

$$m_0 = \min\{m \geq 1 : (m+1)y_n \leq my\}.$$

Then

$$\bigcup_{m=m_0}^{\infty} A_{m,n} = [m_0 y_n, \infty)$$

$$\begin{aligned} \implies & \quad \forall x \in [m_0 y_n, \infty), \quad d(x, \mathcal{K}) \leq \varepsilon_n \\ \implies & \quad \limsup d(x, \mathcal{K}) \leq \varepsilon_n \quad \forall n \geq 1. \end{aligned}$$

Tend $n \rightarrow \infty$. □

Assume now that $\mathcal{K} = \{y_0, y_1, y_2, \dots\}$, where $y_0 = 0$, $y_{n+1} \geq y_n \forall n$ and $y_n \rightarrow \infty$ as $n \rightarrow \infty$.

Since $\mathcal{K} \supseteq \{y_1, 2y_1, 3y_1, \dots, my_1, \dots\}$,

$$\implies \quad 0 < t_{n+1} \equiv y_{n+1} - y_n \leq y_1 \quad \forall n.$$

Consider the set

$$\mathcal{L} = \{t_n; n \geq 1\}.$$

There may be two possibilities:

either $\mathcal{L}$ is infinite

or $\mathcal{L}$ is finite (i.e. t_n take finite number of different values only).

Lemma 9 *If $\mathcal{L}$ is infinite, then $\mathcal{K}$ is dense.*

Proof. Since $0 \leq t_n \leq y_1$, there exists a subsequence $t'_n \rightarrow t$, $t'_n \neq t \forall n$, $t'_n \neq t'_m \forall n \neq m$. Then the proof is similar to that of Lemma 8 (omitted).

Lemma 10 *If $\mathcal{L}$ is finite, then ξ_1 has a lattice distribution.*

Proof. Set $\delta = \min\{t \in \mathcal{L}\}$ and $\delta = t_m$ for some m . Prove that $\frac{t_n}{\delta}$ is an integer for all n . Assume it is not true and $\exists n$:

$$\frac{t_n}{\delta} = k_n + \varepsilon_n; \quad 0 < \varepsilon_n < 1, \quad k_n \geq 1 \text{ — an integer.}$$

$$\implies y_{n+1} - y_n = k_n \delta + \varepsilon_n \delta \equiv k_n (y_{m+1} - y_m) + \varepsilon_n \delta.$$

Note that $y_{n+1} + k_n y_m \in \mathcal{K}$, $y_n + k_n y_{m+1} \in \mathcal{K}$ and

$$0 < (y_{n+1} + k_n y_m) - (y_n + k_n y_{m+1}) = \varepsilon_n \delta < \delta$$

$$\implies \exists l: y_{l+1} - y_l < \delta \implies \text{Contradiction!} \quad \square$$

Next Part