On the ergodicity conditions for stochastically recursive sequences

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The ergodicity criterion for stochastically recursive sequences is given in terms of imbedded subsequences. Several examples are considered.

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1. Introduction

Let $(\mathscr{X}, \mathscr{B}_{\mathscr{X}})$ and $(\mathscr{Y}, \mathscr{B}_{\mathscr{Y}})$ be two measurable spaces, $f: \mathscr{X} \times \mathscr{Y} \to \mathscr{X}$ a measurable function and $\{\xi_n, -\infty < n < \infty\}$, $\{W_n, n \geqslant 0\}$ two sequences of random variables on the probability space $\langle \Omega, \mathscr{F}_{\Omega}, \mathscr{P} \rangle$ with values in $\mathscr{Y}$ and $\mathscr{X}$, correspondingly. We shall say (see [1]) that $\{W_n, n \geqslant 0\}$ is a *stochastically recursive sequence (SRS) with driver* $(\{\xi_n\}, f)$ if it satisfies the relations

$$W_{n+1} = f(W_n, \xi_n), \quad n \geqslant 0. \tag{1}$$

SRS is known to play an important role in queueing analysis. One can find the first steps in the studying of SRS in [2] in the case $\mathscr{X} = \mathbb{R}^d$ when the function f is monotone on the first argument. Borovkov (see [3]) obtained general ergodicity and stability results for SRS based on the ideas of the so-called renovation method. Franken et al. (see, e.g., [4]) proved several results for SRS based on the theory of point processes. The given results were applied for the study of queueing models (see also [5,6] and references in these papers).

We shall suppose in this paper that $(\mathscr{X}, \mathscr{B}_{\mathscr{X}})$ and $(\mathscr{Y}, \mathscr{B}_{\mathscr{Y}})$ are separable metric spaces and $\{\xi_n\}$ is a stationary metrically transitive sequence.

We shall say that a sequence $\{W_n\}$ has *coupling ergodic property (CEP)* if there exists a stationary sequence $\{W^n\}$ such that

$$\mathbf{P}\{W_k = W^k \text{ for each } k \geqslant n\} \to 1 \quad \text{as } n \to \infty. \tag{2}$$

Note that the existence of the stationary sequence of the so-called renovating events (see [3]) is sufficient but not necessary for *CEP* (see also [6,7]). Therefore it is interesting (at least) to formulate the criteria (necessary and sufficient conditions) for *CEP*, and we obtain one of them in this paper. More specifically, we shall demonstrate that if $\{W_{\nu_n}\}$ has *CEP* (where $1 \leqslant \nu_1 < \cdots < \nu_n < \cdots$ is a random sequence of a special form) then $\{W_n\}$ has *CEP* too.

The idea of this criterion is close to the well-known ones in the theory of Markov chains and regenerative processes. For example, assume that W_n is a homogeneous Markov chain with "positive" atom $x \in X$ such that for this x the recurrence time $\tau = \min\{k\colon W_k = x / W_0 = x\}$ has finite mean and G.C.D. $\{n\colon \mathbf{P}\{\tau = n\} > 0\} = 1$. Let $\nu_1 < \nu_2 < \cdots$ be the consecutive recurrence times: $W_{\nu_{k+1}} = \min\{n > W_{\nu_k}\colon W_n = x\}$. The sequence $\{W_{\nu_k}\}$ consists of constants. Particularly, it is stationary and has *CEP*. Therefore the initial sequence $\{W_n\}$ is known to obtain *CEP* (see, e.g., [8, chapter 6]).

2. Definitions and main result

Introduce σ-algebras $\mathscr{F}_n = \sigma\{\xi_k, k \leqslant n\}$, $-\infty < n < \infty$ and $\mathscr{F} = \mathscr{F}_\infty = \sigma\{\xi_k, -\infty < k < \infty\}$. Define the measure-preserving shift transformation U of $\mathscr{F}$-measurable r.v.'s such that $U\xi_n = \xi_{n+1}$, and shift transformation T of $\mathscr{F}$-measurable sets such that $T\{\omega\colon \xi_k(\omega) \in B\} = \{\omega\colon \xi_{k+1}(\omega) \in B\}$, $B \in \mathscr{B}_{\mathscr{Y}}$. We shall use the notation U^n, T^n, $-\infty < n < \infty$ for the iterations of the corresponding transformations (see, e.g., [2]).

Let now $\{A_n = T^n A_0, -\infty < n < \infty\}$, $A_0 \in \mathscr{F}$, $\mathbf{P}\{A_0\} > 0$ be a stationary sequence of events. Introduce random variables $\nu_0 \equiv 0$, $\nu_1 = \min\{n > 0\colon I(A_n) = 1\}$, $\nu_{-1} = \max\{n < 0\colon I(A_n) = 1\}$, $\nu_{k+1} = \min\{n > \nu_k\colon I(A_n) = 1\}$, $\nu_{-k-1} = \max\{n < \nu_{-k}\colon I(A_n) = 1\}$, $k \geqslant 1$; and variables $\mu_k = \nu_k - \nu_{k-1}$, $-\infty < k < \infty$. Note that ν_k are finite a.e. Here $I(A)$ is an indicator of an event A.

Introduce the space

$$\tilde{\mathscr{Y}} = \bigcup_{k \geqslant 1} \{k\} \times \mathscr{Y}^k \equiv \bigcup_{k \geqslant 1} \{(k, y_1, \ldots, y_k);\ y_1, \ldots, y_k \in \mathscr{Y}\}$$

with σ-algebra $\mathscr{B}_{\tilde{\mathscr{Y}}} = \sigma\{\mathscr{B}_{\tilde{\mathscr{Y}},k}, k \geqslant 1\}$, where $\mathscr{B}_{\tilde{\mathscr{Y}},k} = \{D = (k, B), B \in \mathscr{B}_{\mathscr{Y}^k}\}$. Define the sequence

$$\tilde{\xi}_n = \left(\mu_{n+1}, \xi_{\nu_n}, \xi_{\nu_n+1}, \ldots, \xi_{\nu_{n+1}-1}\right), \quad -\infty < n < \infty$$

and measurable function $F\colon \mathscr{X} \times \tilde{\mathscr{Y}} \to \mathscr{X}$ of the form

$$F(x, (k, y_1, \ldots, y_k)) = f_{(k)}(x, y_1, \ldots, y_k), \quad k \geqslant 1,$$

where $f_{(1)} = f$ and

$$f_{(k+1)}(x, y_1, \ldots, y_{k+1}) = f\left(f_{(k)}(x, y_1, \ldots, y_k), y_{k+1}\right) \quad \text{for } k \geqslant 1.$$

The sequence $\tilde{W}_n \equiv W_{\nu_n}$, $n \geqslant 0$, satisfies the recursive relations

$$\tilde{W}_{n+1} = F\left(\tilde{W}_n, \tilde{\xi}_n\right), \quad n \geqslant 0,$$

i.e. it is SRS with the driver $\{\{\tilde{\xi}_n\}, F\}$. Note that $\{\tilde{\xi}_n\}$ may not be a stationary sequence in general. Partially, if $\{\xi_n\}$ is an i.i.d. sequence and events A_n belong to σ-algebras $\sigma\{\xi_{n-1}\}$ then $\{\tilde{\xi}_n, n \geqslant 1\}$ is an i.i.d. sequence.

Assume that (2) takes place for a certain sequence $\{W_n\}$ of $\mathscr{F}$-measurable r.v.'s taking values in $\mathscr{X}$. Therefore we shall say that $\{W_n\}$ *c-converges to* $\{W^n\}$ (and write $W_n \xrightarrow[c]{} W^n$). It follows from (2) that $\lambda \equiv \min\{n \geqslant 0: W_n = W^n$ for all $k \geqslant n\}$ is a.e. finite random variable.

Introduce σ-algebra

$$\tilde{\mathscr{F}}_n' = \sigma\left\{\tilde{\xi}_k, k \leqslant n\right\}; \quad \tilde{\mathscr{F}}' = \tilde{\mathscr{F}}_\infty';$$

$$\tilde{\mathscr{F}}_0 = \left\{E \in \mathscr{F}: E = A_0 \cap C; C \in \tilde{\mathscr{F}}'\right\},$$

probability measure $\mathbf{P}_0(E) \equiv \mathbf{P}_0(A_0 \cap C) = \mathbf{P}(A_0 \cap C)/\mathbf{P}(A_0)$ and define the shift transformation $\tilde{T}$ on the probability space $\langle A_0, \tilde{\mathscr{F}}_0, \mathbf{P}_0\rangle$: if $C = \{\tilde{\xi}_{k_1} \in D_1, \ldots, \tilde{\xi}_{k_l} \in D_l\}, k_1 < \cdots < k_l$, $D_i \in \mathscr{B}_{\mathscr{Y}}$, $1 \leqslant i \leqslant l$, $E = A_0 \cap C$, then $\tilde{T}E = A_0 \cap \tilde{T}C$, where $\tilde{T}C = \{\tilde{\xi}_{k_1+1} \in D_1, \ldots, \tilde{\xi}_{k_l+1} \in D_l\}$. One can define the shift transformation $\tilde{U}$ similarly.

LEMMA 1

$\tilde{T}$ and $\tilde{U}$ are measure-preserving shift transformations.

It follows from lemma 1 that $\{\tilde{\xi}_n\}$ is a stationary sequence on the probability space $\langle A_0, \tilde{\mathscr{F}}_0, \mathbf{P}_0\rangle$.

On the probability space $\langle \Omega, \mathscr{F}, \mathbf{P}\rangle$ define the random variables $W_{n,i} = U^{-i}W_{n+i}$, $i,n \geqslant 0$. Note that for every $i \geqslant 0$ the sequence $\{W_{n,i}, n \geqslant 0\}$ satisfies the equations: $W_{n+1,i} = f(W_{n,i}, \xi_n)$, $n \geqslant 0$. It is clear that if $\{W_n\}$ c-converges to the stationary sequence $\{W^n\}$ then for every $i \geqslant 0$ the sequence $\{W_{n,i}\}$ c-converges to $\{W^n\}$ also.

THEOREM 1

The following two conditions are equivalent:

(1) SRS $\{W_n\}$ has *CEP*;

(2) there exists a stationary sequence of events $\{A_n = T^n A_0\}$, $\mathbf{P}(A_0) > 0$ and (on the probability space $\langle A_0, \tilde{\mathscr{F}}_0, \mathbf{P}_0\rangle$) the stationary sequence of r.v.'s $\{\tilde{W}^n\}$ such that for every $i \geqslant 0$ the sequence

$$\tilde{W}_{n+1,i} = F\left(\tilde{W}_{n,i}, \tilde{\xi}_n\right), \quad n \geqslant 0, \tilde{W}_{0,i} = W_{0,i}$$

c-converges to $\{\tilde{W}^n\}$ on $\langle A_0, \tilde{\mathscr{F}}_0, \mathbf{P}_0\rangle$.

3. Proofs

Proof of lemma 1

We have to prove that the equation

$$\mathbf{P}_0\{\tilde{\xi}_{i_1+l}\in D_1,\ldots,\tilde{\xi}_{i_n+l}\in D_n\}=\mathbf{P}_0\left(\tilde{\xi}_{i_1}\in D_1,\ldots,\tilde{\xi}_{i_n}\in D_n\right) \tag{3}$$

takes place for all integers $n\geqslant 1$, $i_1<\cdots<i_n$, $-\infty<l<\infty$ and for all sets $D_1,\ldots,D_n\in\mathscr{B}_{\mathscr{Y}}$. For the sake of simplicity, we shall obtain (3) for $n=1$, $D_1=D=\{k\}\times B$, $B\in\mathscr{B}_{\mathscr{Y}^k}$, $i_1=0$. The following equations are valid:

$$\begin{aligned}\mathbf{P}_0\{\tilde{\xi}_l\in D\}&=\mathbf{P}\left\{A_0\cap\left\{\mu_{l+1}=k,\,(\xi_{\nu_l},\ldots,\xi_{\nu_l+1})\in B\right\}\right\}/\mathbf{P}\{A_0\}\\&=\sum_{r=l}^{\infty}\mathbf{P}\{A_0\cap\{\nu_l=r,\,\mu_{l+1}=k,\,(\xi_r,\ldots,\xi_{r+k-1})\in B\}\}/\mathbf{P}\{A_0\}\\&=1/\mathbf{P}\{A_0\}\times\sum_{r=l}^{\infty}\mathbf{P}\{A_0\cap\{\nu_{-l}=r,\,\mu_1=k,\,(\xi_0,\ldots,\xi_{k-1})\in B\}\}\\&=\mathbf{P}\{A_0\cap\{\mu_1=k,\,(\xi_0,\ldots,\xi_{k-1})\in B\}\}/\mathbf{P}\{A_0\}\\&=\mathbf{P}_0\{\tilde{\xi}_0\in D\}.\quad\square\end{aligned}$$

Proof of theorem 1.

$\rightarrow$ It is clear that the statement is valid for $A_0=\Omega$.

$\leftarrow$ For any $-\infty<i,l<\infty$ introduce random variables $\nu_{l,i}=U^i\nu_l$, $\mu_{l,i}=U^i\mu_l$ and $\tilde{\xi}_{l,i}=U^i\tilde{\xi}_l$ (particularly, $\nu_{l,0}\equiv\nu_l$, $\mu_{l,0}\equiv\mu_l$ and $\xi_{l,0}\equiv\xi_l$). For any $-\infty<i<\infty$ we can define the probability space $\langle A_i,\,\mathscr{F}_i,\,\mathbf{P}_i\rangle$ similarly to the space $\langle A_0,\,\mathscr{F}_0,\,\mathbf{P}_0\rangle$. On the space $\langle A_i,\,\mathscr{F}_i,\,\mathbf{P}_i\rangle$ we can introduce the shift transformations $\tilde{U}_i^l$ and $\tilde{T}_i^l$, where $\tilde{U}_i\equiv\tilde{U}_i^1$ is such that $\tilde{U}_i\tilde{\xi}_{l,i}=\tilde{\xi}_{l+1,i}$ and $\tilde{U}_i^l$, $\tilde{T}_i^l$ are the iterations of $\tilde{U}_i$, $\tilde{T}_i$, correspondingly.

Let $\tilde{W}^n$ be the stationary sequence defined in theorem 1 and $x\in\mathscr{X}$ be some constant. Denote

$$\tilde{W}^{n,0}=\begin{cases}\tilde{W}^n & \text{on the event } A_0;\\ x & \text{on the event } \overline{A}_0;\end{cases}$$

and $\tilde{W}^{n,i}=U^i\tilde{W}^{n,0}$, $-\infty<n,i<\infty$.

LEMMA 2

Under the conditions of theorem 1 the equation

$$\tilde{W}^{n+1,0}=F\left(\tilde{W}^{n,0},\,\tilde{\xi}_n\right) \tag{4}$$

is valid **P**-a.e. on the event A_0 for every n.

Proof

Without loss of generality, we can prove (4) in the case $n=0$ only. Note that

$$\begin{aligned}
&\mathbf{P}_0\left\{\tilde{W}^{1,0}=F\left(\tilde{W}^{0,0},\ \tilde{\xi}_{0,0}\right)\right\}\\
&\quad\geqslant \mathbf{P}_0\left\{\tilde{U}_0^{-l}\tilde{W}_{l,0}=\tilde{W}^{0,0};\ \tilde{W}^{1,0}=F\left(\tilde{U}_0^{-1}\tilde{W}_{l,0};\ \tilde{\xi}_{0,0}\right)\right\}\\
&\quad=\mathbf{P}_0\left\{\tilde{U}_0^{-l}\tilde{W}_{l,0}=W^{0,0};\ \tilde{U}_0^{-l}\tilde{W}_{l+1,0}=\tilde{W}^{1,0}\right\}\\
&\quad=\mathbf{P}_0\left\{\tilde{W}_{l,0}=\tilde{W}^{l,0};\ \tilde{W}_{l+1,0}=\tilde{W}^{l+1,0}\right\}\\
&\quad=\mathbf{P}_0\left\{\tilde{W}_{k,0}=\tilde{W}^{k,0}\text{ for all }k\geqslant l\right\}\to 1\quad\text{as } l\to\infty.\qquad\square
\end{aligned}$$

Define for $k<l$ the events

$$D_{k,l}=A_k\cap\left(\bigcap_{j=k+1}^{l-1}\bar{A}_j\right)\cap A_l,\ \text{where } \bar{A}_j=\Omega-A_j.$$

LEMMA 3

Under the conditions of theorem 1

$$\tilde{W}^{n+1,k}=\tilde{W}^{n,l} \tag{5}$$

P-a.e. on the event $D_{k,l}$ for every $n,\ k<l$.

Proof

We shall prove (5) in the case $k=0,\ n=0$. Note that

$$\begin{aligned}
&\mathbf{P}\left\{D_{0,l}\cap\{\tilde{W}^{1,0}=\tilde{W}^{0,l}\}\right\}\\
&\quad\geqslant \mathbf{P}\left\{D_{0,l}\cap\left\{\tilde{W}^{0,l}=\tilde{U}_l^{-i-1}\tilde{W}_{i+1,l};\ \tilde{W}^{0,0}=\tilde{U}_0^{-i}\tilde{W}_{i,0};\right.\right.\\
&\qquad\left.\left.\tilde{U}_l^{-i-1}\tilde{W}_{i+1,l}=\tilde{U}_0^{-i}\tilde{W}_{i,0}\right\}\right\}
\end{aligned}$$

and $\tilde{U}_l^{-i-1}\tilde{W}_{i+1,l}=\tilde{U}_0^{-i}\tilde{W}_{i,0}$ **P**-a.e. on the event $D_{0,l}$ for every $i\geqslant 0$. Therefore the right hand side of the previous inequality converges to $\mathbf{P}\{D_{0,l}\}$ as $i\to\infty$. $\square$

Introduce the stationary sequence $\{W^n\}$:

$$W^0=\begin{cases}\tilde{W}^{0,0} & \text{on the event } A_0;\\ f_{(k)}\left(\tilde{W}^{-k,0},\ \xi_{-k},\ldots,\xi_{-1}\right) & \text{on the event } A_{-k}\cap\left(\bigcap_{j=k+1}^{0}\bar{A}_j\right),\ k\geqslant 1;\end{cases}$$

$$W^n=U^nW^0,\quad -\infty<n<\infty.$$

LEMMA 4

Under the conditions of theorem 1

$$W^{n+1}=f(W^n,\ \xi_n)$$

P-a.e. for every n.

Proof

It is clear that

$$W^1=f_{(k+1)}\left(\tilde{W}^{-k,0},\ \xi_{-k},\dots,\xi_0\right)=f\left(W^0,\ \xi_0\right)$$

P-a.e. on the event $\bar{A}_1$ and

$$W^1=\tilde{W}^{1,0}=\tilde{W}^{-k,1}\equiv f_{(k+1)}\left(\tilde{W}^{-k,0},\ \xi_{-k},\dots,\xi_0\right)$$
$$=f\left(f_{(k)}\left(\tilde{W}^{-k,0},\ \xi_{-k},\dots,\xi_{-1}\right),\ \xi_0\right)\equiv f\left(W^{0,0},\ \xi_0\right)$$

P-a.e. on the event $D_{-k,1}$ for $k\geqslant 0$. □

Now we can perform the concluding step of the proof. Define for $k\geqslant 0$ the events

$$E_k=A_k\cap\left(\bigcap_{j=0}^{k-1}\bar{A}_j\right).$$

For every $\epsilon>0$ let $N\equiv N(\epsilon)>0$ be such that $\mathbf{P}\{\bigcup_{i=0}^{N}A_i\}\geqslant 1-\epsilon$. Then

$$\begin{aligned}\mathbf{P}\{W_k=W^k \text{ for all } k\geqslant n\}&=\mathbf{P}\{W_n=W^n\}\\&\geqslant\mathbf{P}\left\{\{W_n=W^n\}\cap\left\{\bigcup_{i=0}^{N}A_i\right\}\right\}\\&=\sum_{k=0}^{N}\mathbf{P}\{\{W_n=W^n\}\cap E_k\}\\&=\sum_{k=0}^{N}\mathbf{P}\left\{\left\{U^{-k}W_n=W^{n-k}\right\}\cap T^{-k}E_k\right\},\end{aligned}$$

and for $l=n-k$

$$\mathbf{P}\left\{\left\{U^{-k}W_{l+k}=W^l\right\}\cap T^{-k}E_k\right\}\geqslant\mathbf{P}\left\{\{\gamma_k\leqslant l\}\cap T^{-k}E_k\right\}\to\mathbf{P}\left\{T^{-k}E_k\right\}=\mathbf{P}\{E_k\}$$

as $l\to\infty$. Here $\gamma_k=\min\{\nu_n\colon W_{\nu_n,k}\equiv\tilde{W}_{n,k}=\tilde{W}^n\}$. Therefore

$$\liminf_{n\to\infty}\mathbf{P}\{W_k=W^k \text{ for all } k\geqslant n\}\geqslant 1-\epsilon$$

for every $\epsilon>0$. The proof of theorem 1 is completed. □

4. Examples

EXAMPLE 1

Let L be a non-negative integer. We shall say (see [3]) that the event $A_n \in \mathscr{F}$ is *renovating* on the interval $(n-L, n)$ for SRS $\{W_n\}$ if there exists a measurable function g such that

$$W_{n+1} = g(\xi_{n-L}, \dots, \xi_n) \tag{6}$$

P-a.e. on the event A_n. The sequence of events $\{A_n, n \geqslant n_0\}$ is the *renovating sequence* for SRS $\{W_n\}$ if there exist a non-negative integer $L \leqslant n_0$ and a function g such that (6) takes place for every $n \geqslant n_0$. Borovkov ([3]) obtained the following result:

THEOREM 2

If $\{A_n = T^n A_0, n \geqslant n_0\}$, $\mathbf{P}\{A_0\} > 0$ is a stationary renovating sequence for SRS $\{W_n\}$ then the sequence $\{W_n\}$ c-converges to a stationary one.

Note that theorem 2 is a simple corollary of theorem 1. Indeed, it is clear that the sequence $\tilde{W}^n = g(\xi_{\nu_n - L}, \dots, \xi_{\nu_n})$ is stationary on the probability space $\langle A_0, \mathscr{F}_0, \mathbf{P}_0 \rangle$ and $\tilde{W}_{n_0,i} = \tilde{W}^{n_0}$ $\mathbf{P}_0$-a.e. for every $i \geqslant 0$.

EXAMPLE 2

We shall apply theorem 1 to the queueing system with several types of customers. Define a stationary metrically transitive sequence $\{\xi_n \equiv (\tau_n, \sigma_n, S_n)\}$ where τ_n are inter-arrival times, S_n is the service time and σ_n is the type of the nth customer, $\sigma_n \in \{1, 2, 3\}$ a.e. Consider a 2-station open queueing system with 3 types of customers and FCFS discipline. The nth customer has the type i if $\sigma_n = i$. The customers of the first type are served on the first station, the customers of the second type on the second one, the customers of the third type on both stations in parallel. In this system the vectors of virtual waiting times $\{W_n = (W_{n,1}, W_{n,2})\}$ satisfy the recursive relations:

$$W_{n+1} = \begin{cases} (W_{n,1} + s_n - \tau_n, W_{n,2} - \tau_n)^+ & \text{if } \sigma_n = 1; \\ (W_{n,1} - \tau_n, W_{n,2} + S_n - \tau_n)^+ & \text{if } \sigma_n = 2; \\ i(\max(W_{n,1}, W_{n,2}) + S_n - \tau_n)^+ & \text{if } \sigma_n = 3; \end{cases}$$

where $i = (1, 1)$.

Consider the sequence $A_n = \{\sigma_{n-1} = 3\}$. Denote

$$\Phi_{n,1} = \sum_{i=\nu_n}^{\nu_{n+1}-2} S_i \cdot I(\sigma_i = 1),$$

$$\Phi_{n,2} = \sum_{i=\nu_n}^{\nu_{n+1}-2} S_i \cdot I(\sigma_i = 2),$$

$\Phi_n = \max(\Phi_{n,1}, \Phi_{n,2}) + S_{\nu_{n+1}-1}$, where $\{\nu_n\}$ were defined in section 2. It is clear that on the probability space $\langle A_0, \mathscr{F}_0, \mathbf{P}_0 \rangle$ the sequence $\{\Phi_n\}$ is stationary but may be not metrically transitive in general. There exist several versions of sufficient conditions for the metrical transitivity of $\{\Phi_n\}$. We shall consider one of them. For each $n > 0$, introduce σ-algebra $\mathscr{B}_n$ generated sequence $\{\xi_n, \xi_{n+1}, \ldots)$. Denote by $\mathscr{B}$ the tail σ-algebra $\mathscr{B} \equiv \lim_{n\to\infty} \mathscr{B}_n$.

COROLLARY 1

Suppose that

(1) σ-algebra $\mathscr{B}$ is trivial, i.e. it consists of two elements only: Ω and $\emptyset$;

(2) $\mathbf{E}_0\{\Phi_0\} < \mathbf{E}\tau_1/\mathbf{P}\{A_0\}$.

Then there exists a stationary sequence $\{W^n\}$ such that $\{W_n\}$ c-converges to $\{W^n\}$ for any initial condition W_0.

Proof

It is clear that the sequence $\{\tilde{W}_n = (\tilde{W}_{n,1}, \tilde{W}_{n,2})\}$ satisfies the equations $\tilde{W}_{n,1} = \tilde{W}_{n,2}$ a.e. on A_0 for all $n \geqslant 1$. Denote $u_n = \tilde{W}_{n,1}$ and note that $u_{n+1} = F(u_n, \xi_n)$, where function F has the form $F(u_n, \xi_n) = \max\{\Psi_{n,1}, \Psi_{n,2}, u_n + \Phi_n - T_n\}$. Here

$$T_n = \sum_{i=\nu_n}^{\nu_{n+1}} \tau_i;$$

$$\Psi_{n-1,j} = \max\{0, S_{\nu_n-1,3} - \tau_{\nu_n-1}, S_{\nu_n-1,j} + S_{\nu_n-2,j} - \tau_{\nu_n-1} - \tau_{\nu_n-2}, \ldots, S_{\nu_n-1,j} + S_{\nu_n-2,j} + \cdots + S_{\nu_{n-1},j} - \tau_{\nu_n-1} - \tau_{\nu_n-2} - \cdots - \tau_{\nu_{n-1}}\}$$

and $S_{k,j} = S_k \cdot I(\sigma_k = j)$, $j = 1, 2, 3$. Denote $\Psi_n = \max\{\Psi_{n,1}, \Psi_{n,2}\}$, $v_n = u_n - \Psi_{n-1}$, $\Gamma_n = \Phi_n - T_n - \Psi_n$. Then

$$v_{n+1} = \max(0, v_n + \Psi_{n-1} + \Gamma_n), \quad n \geqslant 0.$$

Under condition (1) the sequence $\{\Psi_{n-1} + \Gamma_n\}$ satisfies SLLN. Therefore the ergodicity condition for $\{v_n\}$ has the form

$$\begin{aligned}\mathbf{E}_0\{\Psi_{n-1} + \Gamma_n\} &= \mathbf{E}_0\{\Phi_n - T_n\} \\ &= \mathbf{E}_0\{\Phi_n\} - \mathbf{E}_0\{T_n\} < 0,\end{aligned}$$

where $\mathbf{E}_0\{T_0\} = \mathbf{E}\tau_0 \cdot \mathbf{E}_0\{\nu_0\} = \mathbf{E}\tau_0/\mathbf{P}\{A_0\}$. □

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