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STOCHASTICALLY RECURSIVE SEQUENCES AND THEIR GENERALIZATIONS

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Abstract

The paper deals with the stochastically recursive sequences $\{X(n)\}$ defined as the solutions of equations $X(n+1) = f(X(n), \xi_n)$ (where ξ_n is a given random sequence), and with random sequences of a more general nature, named recursive chains. For those the theorems of existence, ergodicity, stability are established, the stationary majorants are constructed. Continuous-time processes associated with ones studied here are considered as well.

Key words and phrases: stochastically recursive sequence; recursive chain; generalized Markov chain; renovating event; coupling-convergence; ergodicity; stability; rate of convergence; stationary majorants; boundedness in probability; processes admitting embedded stochastically recursive sequences.

CHAPTER 1. INTRODUCTION

The main objects of study in this paper are random sequences $\{X(n)\}$ of two types: (a) the stochastically recursive sequences (SRS) defined by recursive relations of the form

$$X(n+1) = f(X(n), \xi_n), \tag{1}$$

where f is a given function and $\{\xi_n\}$ a stationary sequence; (b) sequences $\{X(n)\}$, named in the paper the *recursive chains* (RC), which are characterized by the fact that not the value of $X(n+1)$ itself but only its conditional distribution with respect to the entire prehistory is a function of $(X(n), \xi_n)$. SRS were studied in [1]-[5] and other works, RC were introduced in [6], but they are treated systematically for the first time in the present paper. The sequences of types (a) and (b) are frequently encountered in applications (see, e.g., [1]-[5]), and both types are more general than the Markov chains (MC) (it is evident, that RC belong to the latter class for $\xi_n \equiv$ *const*, and SRS - for independent $\{\xi_n\}$).

Comparing the study of SRS carried out in [2], [4] with the MC ergodic theory, one can hardly find anything in common from the first glance. The more so that the starting points for the study of SRS and MC were completely different. However, it appears that there are some common features. Moreover, the ideas of artificial regeneration construction for MC, introduced in [7], [8], can be effectively used in the theory of SRS and RC, so that the general ergodicity conditions for MC and for the processes defined above can be made rather close both formally and in essence. These conditions are rather general, as well as those for MC, and imply convergence in a very strong form, which yields also the convergence in total variation.

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The paper consists of 7 chapters which are subdivided (but for Chapter 1 and Chapter 6) into sections.

In Chapter 2 the ergodicity conditions for MC and SRS are introduced: in Section 1 a well-known ergodicity criterion is formulated for MC satisfying the Harris condition; in Section 2 the ideas of the renovations methods [2], [4] are developed, which enable one to establish the ergodicity of SRS; in Section 3 the ergodicity criteria for SRS with non-stationary driver are introduced; in Section 4 it is demonstrated that for MC the ergodicity conditions formulated in Section 1 are equivalent in a certain sense to the "renovation" conditions; and finally some simple estimates of the convergence rate in the ergodicity theorem for SRS are presented in Section 5.

In Chapter 3 we define RC, a more general object than MC and SRS. In Section 1 the definitions are given and the problems of construction, existence and uniqueness of RC are discussed. In Section 2 it is shown that a RC can be represented as a SRS with an "extended" driver.

In Chapter 4 the ergodicity problems for RC are discussed. In Section 1 the general ergodicity criteria for RC are formulated. In Section 2 the ergodicity criteria for RC with non-stationary drivers are introduced. Some conditions, which are sufficient for RC ergodicity and are based on a "mixing" condition for the driver, are presented in Section 3.

In Chapter 5 the study of ergodicity proceeds along with the study of boundedness in probability for the random sequences. A modification of the ergodicity conditions , which is connected with the specification of the renovating events structure, is considered in Section 1. In Sections 2 and 3 certain conditions are presented, which are sufficient for construction of the so-called V-inducing events (which are components of the renovating events) for various phase spaces. Section 4 deals with some conditions that ensure boundedness in probability of random sequences. In Section 5 another way is suggested to obtain sufficient conditions for the existence of V-inducing events.

In Chapter 6 the theorem on stability of RC is proved.

Chapter 7 considers the stochastic processes in continuous and discrete time, for which the ergodicity problem may be reduced in a certain sense to the same problem for "imbedded" RC. In Section 1 the processes admitting embedded RC are defined. In Section 2 the ergodicity conditions for such processes are formulated in the case when the elements of the driver are independent. Section 3 introduces the notion of a process admitting imbedded MC, and an ergodicity criterion for such processes is presented in the case when the imbedded MC is Harris ergodic. In Section 4 the ergodicity conditions are formulated for general processes admitting imbedded RC. Finally some examples of processes admitting imbedded RC are given in Section 5.

Enumeration of theorems, lemmas, formulae, etc. is independent in each chapter. Double numbers are used to refer to contents of another chapter.

CHAPTER 2. ERGODICITY CONDITIONS FOR MARKOV CHAINS AND STOCHASTICALLY RECURSIVE SEQUENCES

1. Markov chains

Let $(\mathbf{X}, \mathbf{B_X})$ be an arbitrary measurable space and let $X = \{X(n) = X(x,n);\ n \geq 0\}$ be an **X**-valued homogeneous Markov chain (MC) with the initial state $X(x,0) = x \in \mathbf{X}$. Fairly general ergodicity conditions of MC were established in [7]-[11]. There exist several closely resembling versions of these conditions. We dwell on one of them introduced in [6], [12], [13].

For some set $V \in \mathbf{B}_{\mathbf{X}}$ denote $\tau_V(x) = \min\{i \geq 1 : X(x,i) \in V\}$. Suppose that there exist a set $V \in \mathbf{B}_{\mathbf{X}}$, a probability measure φ on $(\mathbf{X}, \mathbf{B}_{\mathbf{X}})$, a number $p \in (0,1)$, and a non-negative integer $m \geq 0$ such that

(I) $$\mathbf{P}(\tau_V(x) < \infty) = 1 \text{ for any } x \in \mathbf{X};\ \sup_{x \in V} \mathbf{E}\,\tau_V(x) < \infty;$$

(II) $$\inf_{x \in V} \mathbf{P}(X(x, m+1) \in B) \geq p \cdot \varphi(B) \text{ for any } B \in \mathbf{B}_{\mathbf{X}}.$$

Condition (I) means "uniform" positive recurrence of the set V and MC irreducibility. Condition (II) is a "mixing" condition. It is expressed in terms of the local characteristics and in this sense it is final. In "practical" problems Condition (I) usually requires additional consideration.

For the ergodicity of a MC we shall need also the non-periodicity property. If Conditions (I) - (II) are satisfied, then this property may be expressed in terms of m, φ, V, τ_V.

Let $n_1, n_2, \ldots$ be the integer numbers for which $\mathbf{P}(\tau_V(\varphi) = n_i) > 0$, where $\tau_V(\varphi) = \min\{i \geq 1 : X(\varphi, i) \in V\}$; let $X(\varphi, i)$ be a MC with a random initial value distributed according to φ.

(III) (Non-periodicity condition). There exists a number $l > 0$ such that the greatest common divisor of the set $(m + n_1 + 1, m + n_2 + 1, \ldots, m + n_l + 1)$ is equal to one. (This condition is satisfied if $m = 0$, $\varphi(V) > 0$).

Denote $\mathbf{P}(x, B) = \mathbf{P}(X(x,1) \in B)$, $B \in \mathbf{B}_{\mathbf{X}}$.

Theorem 1. *Let Conditions* (I)-(III) *be satisfied. Then there exists a stationary MC* $\{X^n\}$ *with transition probability* $\mathbf{P}(x, B)$ *defined on the same probability space with* X, *which is independent of* $X(0)$ *and such that for each* $x \in \mathbf{X}$

$$\mathbf{P}(X(x,k) = X^k \text{ for all } k \geq n) \to 1 \tag{1}$$

as $n \to \infty$.

The sets $D_n = \{X(x,k) = X^k$ for all $k \geq n\}$ here and in the sequel are not necessarily events (i.e. belong to $\mathbf{B}_{\mathbf{X}}$). In fact in Theorem 1 and a series of consecutive assertions it is proved that there exist events $C_n \subset D_n$ such that $\mathbf{P}(C_n) \to 1$. If $\mathbf{X}$ is a separable metric space and $\mathbf{B}_{\mathbf{X}}$ contains the Borel σ-algebra, then $D_n \in \mathbf{B}_{\mathbf{X}}$.

Here and whenever needed, we assume the σ-algebra $\mathbf{B}_{\mathbf{X}}$ to be countably-generated. (Recall that a σ-algebra $\mathbf{B}_{\mathbf{X}}$ is called countably-generated if it is generated by a countable collection of sets from X). It is evident that the σ-algebra of Borel sets in $\mathbf{X} = \mathbf{R}^d$ is countably-generated. The Borel σ-algebra in any metric separable space is also countably-generated. Therefore the assumption that a σ-algebra is countably-generated is not too restrictive. For countably-generated σ-algebras the function $\mathbf{P}(x, B)$ may be considered to be a measure, whose values are measurable with respect to x (see [11]).

Relation (1) implies necessarily that the distribution of X^0

$$\pi(B) = \mathbf{P}(X^0 \in B), \quad B \in \mathbf{B}_{\mathbf{X}},$$

is an invariant measure:

$$\pi(B) = \int_{\mathbf{X}} \pi(dx)\, \mathbf{P}(x, B), \quad B \in \mathbf{B}_{\mathbf{X}},$$

and the convergence in total variation occurs,

$$\sup_{B \in \mathbf{B}_{\mathbf{X}}} |\, \mathbf{P}(X(x, n) \in B) - \pi(B)\,| \to 0 \tag{2}$$

as $n \to \infty$.

The following assertion, converse to Theorem 1, is also justified.

Theorem 2. *If* (2) *or* (1) *holds for each* $x \in \mathbf{X}$*, then there exist a set* V*, a probability measure* φ *and numbers* p, m*, such that Conditions* (I)-(III) *are satisfied.*

The most essential points of the assertions of Theorems 1,2 have been proved in [10], [11] (see also [6]).

2. Stochastically recursive sequences

Assume that another measurable space $(\mathbf{Y}, \mathbf{B}_{\mathbf{Y}})$ with measure $\mathbf{P}$ is considered along with $(\mathbf{X}, \mathbf{B}_{\mathbf{X}})$, and let $\{\xi_n\}_{n=0}^{\infty}$ be a random sequence with values in $(\mathbf{Y}, \mathbf{B}_{\mathbf{Y}})$. Thereto, let a measurable function $f : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ be specified on $(\mathbf{X} \times \mathbf{Y}, \mathbf{B}_{\mathbf{X}} \times \mathbf{B}_{\mathbf{Y}})$.

Definition 1. We shall say that $\{X(n)\}$ is a stochastically recursive sequence (SRS) *with driver* $\{\xi_n\}$ if it satisfies relations

$$X(n+1) = f(X(n), \xi_n)$$

for all $n \geq 0$. For the sake of simplicity we shall assume the value $X(0)$ to be fixed and non-random (or to be random and independent of $\{\xi_n\}$).

The construction of the distribution of the sequence $\{X(n), \xi_n\}$ on $((\mathbf{X} \times \mathbf{Y})^{\infty}$, $(\mathbf{B}_{\mathbf{X}} \times \mathbf{B}_{\mathbf{Y}})^{\infty})$ is carried out using finite-dimensional distributions similarly to the construction of the distribution of MC X on $(\mathbf{X}^{\infty}, \mathbf{B}_{\mathbf{X}}^{\infty})$ from the transition function $\mathbf{P}(x, B) = \mathbf{P}(X(x, 1) \in B)$ with evident alterations. The finite-dimensional distributions $\{(X(0), \xi_0), \ldots, (X(k), \xi_k)\}$ are defined according to the relations

$$\mathbf{P}(X(l) \in A_l,\ \xi_l \in B_l;\ l = 0, \ldots, k) =$$

$$= \int_{B_0} \ldots \int_{B_k} \mathbf{P}(\xi_l \in dy_l;\ l = 0, \ldots, k) \cdot \prod_{l=1}^{k} I(f_l(X(0), y_0, \ldots, y_l) \in A_l),$$

where $f_1(x, y_0) = f(x, y_0)$; $f_l(x, y_0, \ldots, y_l) = f(f_{l-1}(x, y_0, \ldots, y_{l-1}), y_l)$.

The sequence $\{\xi_n\}$ can be considered as specified for any $-\infty < n < \infty$ without loss of generality (for a stationary sequence the required extension may always be realized for $n < 0$ with the help of the Kolmogorov theorem).

It should be noted that at present the term "stochastically recursive sequence" may not be considered conventional. The study of SRS started apparently from [1], where the case $\mathbf{X} = \mathbf{R}^d$ was considered, and the function f was assumed to be monotone in the first variable. In [2], [4], [14] the general ergodicity and stability theorems for SRS were proved, which were based on the notion of the so-called renovating (renewing) events. A series of general constructions and assertions for SRS on the base of the point processes theory is contained in [3], [5], where the term "recursive stochastic equations" was used to designate

SRS. In all the cited papers it was supposed that the sequence $\{\xi_n\}$ is stationary; the results obtained were applied to study multi-server queueing systems.

As already mentioned, SRS is a more general object than MC. Namely, any MC can be represented as a SRS with independent $\{\xi_n\}$ (for details see Chapter 3 or [19]).

Define the σ-algebras $\mathbf{F}^{\xi}_{l,n} = \sigma\{\xi_k ; l \le k \le n\}$; $\mathbf{F}^{\xi}_{n} = \sigma\{\xi_k ; k \le n\} = \mathbf{F}^{\xi}_{-\infty,n}$; $\mathbf{F}^{\xi} = \sigma\{\xi_k ; -\infty < k < \infty\} = \mathbf{F}^{\xi}_{-\infty,\infty}$.

Definition 2. We shall say that an event $A \in \mathbf{F}^{\xi}_{n+m}$, $m \ge 0$, is *renovating (renewing) on the interval* $[n, n+m]$ *for SRS* $\{X(n)\}$ if there exists a measurable function $g : \mathbf{Y}^{m+1} \to \mathbf{X}$ such that on the set A (i.e., for $\omega \in A$)

$$X(n+m+1) = g(\xi_n, \dots, \xi_{n+m}). \tag{3}$$

It is evident that the relations of the form

$$X(n+m+k+1) = g_k(\xi_n, \dots, \xi_{n+m+k})$$

are valid for all $k \ge 0$ and for $\omega \in A$, where the function g_k depends on the specified arguments only and is determined by the event A .

A sequence of events $\{A_n\}$, $A_n \in \mathbf{F}^{\xi}_{n+m}$ (where an integer $m \ge 0$ is fixed) is said to be *renovating for the sequence* $\{X(n)\}$ if there exists an integer $n_0 \ge 0$ such that relation (3) holds for any $n \ge n_0$ on the event A_n with the same function g .

The objects of our basic interest are "positive" renovating events, i.e., the events with positive probability $\mathbf{P}(A_n) > 0$.

The simplest example of a renovating event is hitting a fixed point x_0 by $X(n)$: $A_n = \{X(n) = x_0\}$; here $m = 0$. However such an event may have zero probability. We shall consider below another example, richer in content. More examples of renovating events can be found in [2], [4], [15-17].

Note that in general the event A and thus the function g may depend on the initial value $X(0)$.

In the sequel we shall assume the sequence $\{\xi_n\}$ to be stationary and metrically transitive. The symbol U will denote the measure-preserving shift transformation of $\mathbf{F}^{\xi}$-measurable random variables generated by $\{\xi_n\}$, so that $U\xi_n = \xi_{n+1}$; and the symbol T will denote the shift transformation of sets (events) from the σ-algebra $\mathbf{F}^{\xi}$:

$$T\{\omega : \xi_j(\omega) \in B_j ; j = 1, \dots, k\} = \{\omega : \xi_{j+1}(\omega) \in B_j ; j = 1, \dots, k\}.$$

The symbols U^n and T^n, $n \ge 0$, will denote respectively the iterations of these transformations (so that $U^1 = U$, $T^1 = T$, while U^0 and T^0 are identical transformations), and the symbols U^{-n}, T^{-n} will denote the transformations inverse to U^n and T^n respectively.

A sequence of events $\{A_k\}$ is said to be *stationary* if $A_k = T^k A_0$ for any k .

Example 1. Consider a real-valued sequence $X(n+1) = (X(n) + \xi_n)^+$, $X(0) = a \geq 0$, where $x^+ = \max(0, x)$ and $\{\xi_n\}$ is a stationary metrically transitive sequence. For any n_0 the events $A_n = T^n A_0$, where

$$A_0 \equiv A_{0,a} = \{\xi_{-1} \leq 0, \xi_{-1} + \xi_{-2} \leq 0, \ldots, \xi_{-1} + \ldots + \xi_{-n_0} \leq 0;$$
$$\xi_{-1} + \ldots + \xi_{-n_0} + \ldots + \xi_{-n_0 - l} \leq -a \text{ for all } l \geq 1\}, \tag{4}$$

form a stationary sequence of renovating events for $m = 0$, $g(y) \equiv y^+$, since for $n \geq n_0$

$$X(n+1) = \xi_n^+ \quad \text{a.s. on } A_n.$$

If we assume that $\mathbf{E}\xi_1 < 0$, the sequence $\{\xi_n\}$ is metrically transitive and, hence, $\xi_{-1} + \ldots + \xi_{-n} \to -\infty$ a.s. for $n \to \infty$, then there exists such a number $n_0 = n_0(a)$, that $\mathbf{P}(A_{n_0}) > 0$.

On the other hand, if we define the events B_n, the number m and the function $g : \mathbf{R}^{m+1} \to \mathbf{R}$ according to the equalities:

$$m = n_0 \; ; \; B_n = T^m A_n \; ; \; g(y_0, \ldots, y_m) \equiv y_m^+ ,$$

then the events $B_n \in \mathbf{F}^{\xi}_{n+m}$ will be renovating for $\{X(n)\}$ on the interval $[n, n+m]$ for any $n \geq n_0 \equiv 0$.

Similar reasoning can be applied to arbitrary renovating events. Therefore in the sequel we shall assume without loss of generality that the number n_0, involved in the definition of a sequence of renovating events, is equal to zero.

The following assertion was proved in [2], [14].

Theorem 3. *Let there exist a stationary sequence* $\{A_n\}$, $\mathbf{P}(A_0) > 0$, *of renovating events for SRS* $\{X(n)\}$. *Then on the same probability space with* $\{X(n)\}$ *a stationary sequence* $\{X^n \equiv U^n X^0\}$ *can be defined, which satisfies the equations* $X^{n+1} = f(X^n, \xi_n)$ *and the relations*

$$\mathbf{P}\{X(k) = X^k \text{ for all } k \geq n\} \to 1 \tag{5}$$

as $n \to \infty$.

Note that if we introduce the measure $\pi(B) = \mathbf{P}(X^0 \in B)$ similarly to Theorem 1, then (5) yields convergence in total variation:

$$\sup_{B \in \mathbf{B}_{\mathbf{X}}} |\mathbf{P}(X(n) \in B) - \pi(B)| \to 0$$

for $n \to \infty$.

As the renovating events A_n (and also the number m and the function g) generally depend on the initial state $X(0) = x \in \mathbf{X}$, the stationary sequence $\{X^n\}$ may also depend on *x*. However, if we assume that for some set $V_0 \in \mathbf{B}_{\mathbf{X}}$ it is possible to find a stationary sequence of events $A_n \in \mathbf{F}^{\xi}_{n+m}$, which is renovating for $\{X(n)\}$ (with the same g) for any $X(0) = x \in V_0$, then the limiting stationary sequence $\{X^n\}$ and, hence, the measure π will not depend on $X(0) \in V_0$. It is clear that if the initial value $X(0)$ be random, $\mathbf{P}(X(0) \in V_0) = 1$, then the sequence $\{X(n)\}$ will also converge to $\{X^n\}$ in the sense of (5).

All the above implies the following assertion.

Let there exist an increasing sequence of sets $\{V_k\}$, $\bigcup_{k=1}^{\infty} V_k = \mathbf{X}$, *such that for any* k *there exists a stationary sequence of events* $\{A_{nk}\}$, *which is renovating* (*with the same function* g_k) *for* $\{X(n)\}$ *with arbitrary* $X(0) = x \in V_k$. *Then the limiting stationary sequence* $\{X^n\}$ *and the measure* π *will not depend on the initial value* $X(0) = x$ *for arbitrary* $x \in \mathbf{X}$.

E x a m p l e 2. Consider the sequence $X(n)$ from Example 1. Since the function $f(x, y) = (x + y)^+$ is monotone increasing, the events A_{nk} defined in (4) for $a = k$ are renovating for $\{X(n)\}$ for any initial value $X(0) \le k$. Hence, one may take $V_k = [0, k]$ for the sets V_k.

R e m a r k 1. One of the many distinctions between SRS and MC consists in the following fact. We know that periodicity and ergodicity are incompatible for MC (in the sense of relations (1), (2)). This is not so for SRS .

E x a m p l e 3. Let $\mathbf{X} = \mathbf{Y} = \{0, 1\}$, $f(x, y) = y$, and $\{\xi_n\}$ be a stationary metrically transitive sequence of the form $\xi_{n+1} = 1 - \xi_n$, where $\mathbf{P}(\xi_0 = 1) = \mathbf{P}(\xi_0 = 0) = 1/2$. It is evident that $\{\xi_n\}$ form a stationary periodic MC. For the SRS considered we obtain $X(n+1) = f(X(n), \xi_n) = \xi_n$ for any $X(0) \in \{0, 1\}$; the conditions of the ergodicity theorem are evidently satisfied. On the other hand, SRS X is 2-periodic in the following sense:

$$\mathbf{P}(X(n+1) = 1 \mid X(n) = 0) = \mathbf{P}(X(n+1) = 0 \mid X(n) = 1) = 1 .$$

The very notion of periodicity for SRS may be substantially wider, both in the sense of deeper dependence on the prehistory and in the sense of lack of determination, characteristic for the equalities presented above. In order to make this point clear, let us modify Example 3. Along with the sequence $\{\xi_n\}$ of Example 3 we shall consider a stationary sequence of independent random variables $\{\eta_n\}$ (e.g., normally distributed), independent of $\{\xi_n\}$, and take the stationary metrically transitive sequence $\{\xi_n, \eta_n\}$ as a driver, so that in this case $\mathbf{Y} = \{0, 1\} \times \mathbf{R}$; $y = (y_1, y_2)$. Set $f(x, y) = y_1 + y_2$; then $\{X(n)\}$ will be an ergodic SRS, but its periodicity will not be deterministic and will concern only distributions.

In order to formulate the assertion converse to Theorem 3, let us consider along with (5) another type of convergence of random sequences. Let us first introduce one more notion.

Let X^0 be a random variable with values in the space $\mathbf{X}$, measurable with respect to σ-algebra $\mathbf{F}^{\xi}$, and $\{X^n = U^n X^0\}$ the stationary sequence constructed on the base of X^0.

Definition 3. We shall say that a SRS $\{X(n)\}$ *coupling-converges* (or *c-converges*) *to* $\{X^n\}$ and write $X(n) \xrightarrow{c} X^n$ if (5) is justified.

One can reformulate Theorem 3 applying this definition in the following form: *if there exists a "positive" stationary sequence of renovating events for SRS* $\{X(n)\}$, *then* $\{X(n)\}$ *coupling-converges to some stationary sequence.*

If we introduce the random variable

$$\mu_0 = \min \{ n \ge 0 : X(n) = X^n \} \equiv \min \{ n \ge 0 : X(k) = X^k \text{ for } k \ge n \},$$

then assertion (5) can also be rewritten in the form

$$\mathbf{P}(\mu_0 < \infty) = 1. \tag{6}$$

For $k = 0, 1, 2, \ldots$ consider along with $\{X(n)\}$ the sequence

$$X_k(n) = U^{-k} X(n+k), \ n \ge -k,$$

obtained applying shifts U^{-k} to the initial sequence $\{X(n)\}$. In other words, $\{X_k(n)\}$ is determined by the initial value $X_k(-k) = X(0)$ at time $(-k)$ and by recursive relations $X_k(n+1) = f(X_k(n), \xi_n)$ for $n \ge -k$. If we denote $\mu_k = \min\{n \ge -k : X_k(n) = X^n\}$, then it is easy to see that $\mu_k = U^{-k}\mu_0 - k$. Hence, the coupling-convergence of the sequence $\{X(n)\}$ implies for any $k \ge 0$ the coupling-convergence of the sequence $\{X_k(n), n \ge -k\}$ to $\{X^n\}$ (or, equivalently, that of the sequence $\{X_k(n), n \ge 0\}$).

Denote by $\mu^0 = \sup\limits_{k \ge 0} \mu_k$ the time at which the coupling with the sequence $\{X^n\}$ is completed for all the sequences $\{\{X_k(n)\}, k \ge 0\}$.

Definition 4. We shall say that the sequence $\{X(n)\}$ strongly coupling-converges to the sequence $\{X^n \equiv U^n X^0\}$ (or *sc-converges*) and write $X(n) \xrightarrow{sc} X^n$ if $\mu^0 < \infty$ a.s.

Note that sc-convergence is stronger than c-convergence, i.e., the a.s. finiteness of the random variable μ_0 does not imply in general the finiteness of μ^0, which is demonstrated by following example.

E x a m p l e 4. Let $\mathbf{X} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers, $X(0) = 0$, and let $\{\xi_n\}$ be a sequence of independent identically distributed random variables with values $1, 2, 3, \ldots$, such that $\mathbf{E}\xi_n = \infty$,

$$f(x, y) = \begin{cases} y & \text{for } x = 0, \\ x - 1 & \text{for } x \ge 2, \\ x & \text{for } x = 1. \end{cases}$$

Assume that $X^n \equiv 1, -\infty < n < \infty$. It is easy to see that the SRS $\{X(n)\}$ c-converges to $\{X^n\}$. Indeed, $\mu_0 = \xi_0 + 1 < \infty$ a.s. On the other hand,

$$X_k(0) = U^{-k} X(k) = U^{-k}(1 + (\xi_0 - k)^+) = 1 + (\xi_{-k} - k)^+.$$

Thus

$$\mu^0 = \min \{ n \ge 0 : (\xi_{-k} - k)^+ = 0 \text{ for all } k \ge n \} =$$

$$= \min \{ n \ge 0 : \xi_{-k} \le k \text{ for all } k \ge n \} = \infty \text{ a.s.}$$

Let us present also another version of the definition of sc-convergence. Denote $\nu = \min\{n \ge 0 : U^{-k} X(k) = X^0$ for all $k \ge n\}$. It is easily seen that the distributions of the random variables μ^0 and ν coincide. Indeed,

$$\mathbf{P}(\mu^0 \le n) = \mathbf{P}(X_k(n) = X^n \text{ for all } k \ge 0) =$$

$$= \mathbf{P}(X_{k+n}(0) = X^0 \text{ for all } k \ge 0) =$$

$$= \mathbf{P}(X_l(0) = X^0 \text{ for all } l \ge n) = \mathbf{P}(\nu \le n) .$$

So the following theorem is true (see [4], [18]).

Theorem 4. *The conditions of Theorem* 3 *(viz., existence of the stationary sequence of renovating events* $\{A_n\}$ *with* $\mathbf{P}(A_n) > 0$ *) are necessary and sufficient for* sc-*convergence*

$$X(n) \overset{sc}{\to} X^n .$$

For the sake of completeness we present the

Proof of Theorem 4.

N e c e s s i t y. Let the sequence $X(n+1) = f(X(n), \xi_n)$ sc-converge to the stationary sequence $X^{n+1} = f(X^n, \xi_n)$. Choose the number $m \ge 0$ so that the event $A_0 \equiv \{\mu^0 = m+1\} \in \mathbf{F}_m^{\xi}$ has a positive probability. Introduce a function $g: \mathbf{Y}^{m+1} \to \mathbf{X}$,

$$g(y_0, \dots, y_m) = f^{m+1}(X(0), y_0, \dots, y_m) \ ; \ y_0, \dots, y_m \in \mathbf{Y}^{m+1} ,$$

where $f^1 \equiv f$ and $f^{i+1}(x, y_0, \dots, y_i) = f(f^i(x, y_0, \dots, y_{i-1}), y_i)$, $i \ge 1$, are iterations of f . Then the relations

$$X_k(m+1) = X_0(m+1) \equiv g(\xi_0, \dots, \xi_m) , k \ge 0$$

hold a.e. on the set A_0. Hence, for any $n \ge 0$

$$U^n X_k(m+1) = U^n X_0(m+1) = g(\xi_n, \dots, \xi_{n+m}) , k \ge 0$$

a.s. on the set $A_n \equiv T^n A_0$. In particular, for $k = n$

$$U^n X_n(m+1) \equiv X(n+m+1) = g(\xi_n, \dots, \xi_{n+m})$$

a.s. on the set A_n.

S u f f i c i e n c y. Let the sequence $X(n+1) = f(X(n), \xi_n)$ have a stationary sequence of renovating events $\{A_n\}$, $\mathbf{P}(A_0) > 0$. For each n, $-\infty < n < \infty$, define the random variable X^{n+m+1} by the equalities:

$$X^{n+m+1} = g(\xi_n, \dots, \xi_{n+m})$$

on the set A_n and, for $k \ge 1$,

$$X^{n+m+1} = f^k(g(\xi_{n-k}, \dots, \xi_{n-k+m}), \xi_{n-k+m+1}, \dots, \xi_{n+m})$$

on the set $A_{n-k} \cap (\bigcap_{i=1}^{k} \overline{A}_{n-k+i})$.

Since $\mathbf{P} (\bigcup_{i=0}^{\infty} A_{n-i}) = 1$ (see [2]), the random variables $\{ X^n \}$ are defined a.s. By the construction, the sequence $\{ X^n \}$ is stationary. Moreover, the stationarity of $\{ A_n \}$ implies that for all $n, k \geq 0$ the following equalities hold a.s. on the set A_n:

$$X_k (n + m + 1) = g (\xi_n , \dots , \xi_{n+m}).$$

Thus

$$\mathbf{P} (\mu^0 \leq n) \geq \mathbf{P} (\bigcup_{i=0}^{n-1} A_i) \to 1 \text{ for } n \to \infty .$$

Finally, it follows from the type of convergence and the equalities $X_k (n + 1) = f (X_k (n) , \xi_n)$ that the elements of the sequence $\{ X^n \}$ are connected by recursive relations $X^{n+1} = f (X^n , \xi_n)$. The proof of the theorem is completed.

A natural question arises. When are the c-convergence and sc-convergence equivalent, i.e., when does $\mu_0 < \infty$ a.s. imply $\mu^0 < \infty$ a.s.? There exist some rather simple sufficient conditions, providing for the required equivalence, which are based on the monotonicity of the function f in the first variable; for instance, queueing systems in series possess this property.

Lemma 1. *Suppose that a partial order relation ($\leq$) is defined on the space* $\mathbf{X}$ *, and the function f and the initial condition* $X(0) = x$ *possess the following monotonicity properties:*

1) $f (x , y) \geq x$ *for each* $y \in \mathbf{Y}$,

2) *if* $x_1 , x_2 \in \mathbf{X}$, and $x_1 \leq x_2$, *then* $f (x_1 , y) \leq f (x_2 , y)$ *for all* $y \in \mathbf{Y}$.

Then $\mu_0 = \mu^0$ *a.s.*

Proof of Lemma 1 is practically obvious. Indeed, it follows from the monotonicity properties that

$$X (0) = x \leq U^{-1} X (1) \leq \dots \leq U^{-n} X (n) \leq U^{-n-1} X (n+1) \leq \dots \leq X^0 \text{ a.s.}$$

Thus

$$\mathbf{P} (\mu_0 \leq n) = \mathbf{P} (X (n) = X^n) = \mathbf{P} (U^{-n} X (n) = X^0) =$$

$$= \mathbf{P} (U^{-i} X (i) = X^0 \text{ for all } i \geq n) = \mathbf{P} (\nu \leq n) = \mathbf{P} (\mu^0 \leq n).$$

These equalities and the relation $\mu_0 \leq \mu^0$ a.s. imply the equality $\mu_0 = \mu^0$ a.s.

3. Ergodicity of SRS with non-stationary drivers

Ergodic theorems similar to Theorem 3 can be also formulated for a non-stationary driver $\{ \zeta_n \}$ which converges to a stationary one. We introduce two versions of such assertions below (Theorems 5, 6).

Theorem 5. *Let there be given a stationary metrically transitive sequence* $\{ \xi_n \}$, *a sequence* $\{ \zeta_n \}$, *and a set* $V_0 \in \mathbf{B}_{\mathbf{X}}$ *such that*

1) *there exists a stationary sequence of events* $A_n \in \mathbf{F}^{\xi}_{n+m}$ *, which is renovating (with the same function* g *) for SRS* $X(n+1) = f(X(n), \xi_n)$ *with arbitrary* $X(0) = x \in V_0$ *;*

2) *the sequence* $\{\zeta_n\}$ *c-converges to* $\{\xi_n\}$*;*

3) *the sequence* $\{Y(n)\}$ *defined by the relations* $Y(n+1) = f(Y(n), \zeta_n)$ *satisfies the condition*

$$\mathbf{P}\Big(\bigcup_{i \ge n} \{Y(i) \in V_0\}\Big) = 1 \text{ for all } n .$$

Let $\{X^n\}$ *be a stationary sequence such that* $\{X(n)\}$ *sc-converges to* $\{X^n\}$ *for an arbitrary initial value* $X(0) \in V_0$ *. Then* $\{Y(n)\}$ *c-converges to* $\{X^n\}$*, and* $\{X^n\}$ *does not depend on* $X(0) \in V_0$ *.*

Proof. Set

$$\gamma = \min\{n \ge 0 : \xi_i = \zeta_i \text{ for all } i \ge n\}.$$

For a given value of $\varepsilon > 0$ one can find a number n_ε such that $\mathbf{P}(\gamma \le n_\varepsilon) \ge 1 - \varepsilon$. Consider for $i \ge n_\varepsilon$ on the set $\{Y(i) \in V_0\}$ the sequence

$$\tilde{X}(i+n+1) = f(\tilde{X}(i+n), \xi_{i+n}) \ ; \ n \ge 0 ,$$

where $\tilde{X}(i) = Y(i)$. According to Theorem 3 and the remarks to follow, the sequence $\{\tilde{X}(n)\}$, $n \ge i$, c-converges to $\{X^n\}$ on the set $\{Y(i) \in V_0\}$. Define on the set $\{Y(i) \in V_0\}$ the random variable $\mu(i) = \min\{n \ge i : \tilde{X}(n) = X^n\}$. One can find a number N_ε such that

$$\mathbf{P}\left(\bigcup_{i=n_\varepsilon}^{N_\varepsilon} \{Y(i) \in V_0\}\right) \ge 1 - \varepsilon .$$

Denote $\mu_0 = \min\{n \ge 0 : Y(i) = X^i \text{ for } i \ge n\}$. Then for sufficiently large n

$$\mathbf{P}(\mu_0 > n) \le \varepsilon + \mathbf{P}(\gamma \le n_\varepsilon, \mu_0 > n) \le$$

$$\le 2\varepsilon + \sum_{i=n_\varepsilon}^{N_\varepsilon} \mathbf{P}(\gamma \le n_\varepsilon ; Y(i) \in V_0 ; \mu(i) > n) ;$$

and the right-hand side goes to 2ε as $n \to \infty$. The choice of $\varepsilon > 0$ being arbitrary, the assertion of Theorem 5 is proved.

Theorem 5 can be generalized.

Theorem 6. *Let there be given a stationary metrically transitive sequence* $\{\xi_n\}$*, a sequence* $\{\zeta_n\}$*, and a non-decreasing sequence of sets* $V_k \in \mathbf{B}_{\mathbf{X}}$ *such that*

1) *for each* $k = 1, 2, \ldots$ *there exists a stationary sequence of events* $A_{n,k}$ *, which are renovating (with the same parameter* m_k *and function* g_k *) for the sequence* $X(n+1) = f(X(n), \xi_n)$ *with arbitrary* $X(0) = x \in V_k$ *;*

2) *the sequence* $\{\zeta_n\}$ *c-converges to* $\{\xi_n\}$ *.*

Denote by $\{X^n\}$ *the stationary sequence such that* $\{X(n)\}$ *sc-converges to* $\{X^n\}$ *for some initial value* $X(0)=x\in\mathbf{X}$. *Then* $\{X^n\}$ *does not depend on* $X(0)$, *and the sequence* $Y(n+1)= f(Y(n),\zeta_n)$ *c-converges to* $\{X^n\}$ *for any (possibly, random) initial value* $Y(0)\in\mathbf{X}$.

Proof. Denote, as before,

$$\gamma=\min\{n\ge 0:\xi_i=\zeta_i \text{ for all } i\ge n\}.$$

Let us find, for an arbitrary sufficiently small value of $\varepsilon>0$, a number n_ε such that $\mathbf{P}(\gamma\le n_\varepsilon)\ge 1-\varepsilon$, and define the number k_ε so that $\mathbf{P}(Y(n_\varepsilon)\in V_{k_\varepsilon})\ge 1-\varepsilon$. Consider the sequence

$$\tilde{X}(n_\varepsilon+l+1)=f(\tilde{X}(n_\varepsilon+l),\xi_{n_\varepsilon+l}),\ l\ge 0,$$

where

$$\tilde{X}(n_\varepsilon)=\begin{cases} Y(n_\varepsilon), & \text{for } Y(n_\varepsilon)\in V_{k_\varepsilon},\\ y_0, & \text{else},\end{cases}$$

and $y_0\in V_{k_\varepsilon}$ is an arbitrary fixed element.

In accordance with Theorem 3 and the remarks to follow, the sequence $\{\tilde{X}(n)\}$ c-converges to $\{X^n\}$. Thus for $n\ge n_\varepsilon$

$$\mathbf{P}(Y(l)=X^l \text{ for all } l\ge n)\ge$$

$$\ge\mathbf{P}(Y(l)=\tilde{X}(l)=X^l \text{ for all } l\ge n;\gamma\le n_\varepsilon;Y(n_\varepsilon)\in V_{k_\varepsilon})\ge$$

$$\ge\mathbf{P}(\tilde{X}(l)=X^l \text{ for all } l\ge n)-2\varepsilon,$$

and the right-hand side goes to $1-2\varepsilon$ as $n\to\infty$. The choice of $\varepsilon>0$ being arbitrary, the theorem is proved.

Let us consider one more particular case of SRS with a non-stationary driver, for which it is possible to obtain a stronger assertion, viz., to establish sc-convergence instead of c-convergence.

Assume, as before, that $X(n+1)=f(X(n),\xi_n)$ is a SRS with the driver $\{\xi_n\}$, $(\mathbf{Z},\mathbf{B_Z})$ is some measurable space, and $F:\mathbf{Z}\times\mathbf{X}\to\mathbf{Z}$ is a measurable function. Define the SRS

$$Z(n+1)=F(Z(n),X(n)),\ Z(0)=const,$$

assuming values in the space $(\mathbf{Z},\mathbf{B_Z})$ and driven by $\{X(n)\}$. Note that for each n the random variable $Z(n+1)$ is measurable with respect to the σ-algebra $\mathbf{F}_n^\xi$.

In this setting it is possible to prove

Theorem 7. *Suppose that*

1) *the conditions of Theorem* 3 *are satisfied for SRS* $X(n)$;

2) *there exist a non-negative integer M and a stationary "positive" sequence of events* $D_n\in\mathbf{F}_{n+M}^\xi$, *which are renovating for SRS* $Z(n+1)=F(Z(n),X(n))$, *i.e., for some function* $G:\mathbf{X}^{M+1}\to\mathbf{Z}$ *the equality*

$$Z(n+M+1) = G(X(n), \dots, X(n+M))$$

holds for all $n \geq 0$ *and* $\omega \in D_n$.

Then the sequence $\{Z(n)\}$ *sc-converges to some stationary sequence* $\{Z^n\}$ *, satisfying relations* $Z^{n+1} = F(Z^n, X^n)$ *, where* $\{X^n\}$ *is the "sc-limit" of* $\{X(n)\}$.

Proof. Define the random variable

$$\mu^0 = \min\{n \geq 0 : U^{-l} X(i+l) = X^i \text{ for all } l \geq 0, i \geq n\}.$$

Let us find a number $i_0 > 0$ such that

$$\mathbf{P}(\mu^0 \leq i_0) \geq 1 - \mathbf{P}(D_0)/2.$$

Denote $C_0 = \{U^{-i_0}\mu^0 \leq i_0\}$; $C_n = T^n C_0$; $E_n = D_n \cap C_n$.

As soon as $\mathbf{P}(E_n) \geq \mathbf{P}(D_0)/2 > 0$, the events E_n form a stationary "positive" renovating sequence for $\{Z(n)\}$; thereto the equality

$$Z(n+M+1) = G(X^n, \dots, X^{n+M}) = U^{-l} Z(n+M+l+1)$$

holds for any $\omega \in E_n$, $n \geq i_0$, $l \geq 0$, and

$$Z(j+1) = F(Z(j), X^j) = U^{-l} Z(j+l+1)$$

for any $j > n+M$ and $l \geq 0$.
The further reasoning is a verbatim copy of the proof of the first part of Theorem 4.

4. Juxtaposition of ergodicity conditions for MC and SRS

Theorem 8. *If the* σ*-algebra* $\mathbf{B}_{\mathbf{X}}$ *is countably-generated, then each MC is SRS, and its trajectory can be represented in the form*

$$X(n+1) = f(X(n), \xi_n), \quad X(0) = x, \tag{7}$$

for a suitable function f and i.i.d. real-valued $\{\xi_n\}$. *If Conditions* (I)-(III) *are satisfied for a MC* $\{X(n)\}$ *and the initial value* $x \in \mathbf{X}$ *is such that* $\mathbf{E}\tau_V(x) < \infty$ *, then the sequence* $\{X(n)\}$ *sc-converges to some stationary sequence* $\{X^n\}$ *which satisfies* (7).

Proof. The first part of the assertion of Theorem 8 is a particular case of Theorem 3.1 proved in the next chapter (see also [19]). Let us prove the second part of the theorem. If Conditions (I)-(II) hold, the approach introduced in [7], [8] enables one to expand MC so that it acquires regeneration points. If we assume $\{X(n)\}$ to be already an expanded MC, then Conditions (I)-(II) are transformed into the following ones:

1) there exist a set V and a probability measure φ such that $\mathbf{P}(y, B) = \varphi(B)$ for each $y \in V$, $B \in \mathbf{B}_{\mathbf{X}}$;

2) $\mathbf{E}\tau_V(y) < \infty$ for $y \in V$.

Note that property 1) implies the independence of the distribution of $\tau_V(y)$ from $y \in V$. Thus if we fix a point $y_0 \in V$ and introduce a new function f' determined by the equality

$$f'(y,z) = \begin{cases} f(y,z) & \text{for } y \notin V, \\ f(y_0, z) & \text{for } y \in V, \end{cases}$$

where the function f is defined according to the first part of the theorem for the already "expanded" chain, then all the finite-dimensional distributions of the MC $\{X'(n+1) = f'(X'(n), \xi_n)\}$, $X'(0) = x$ coincide with the corresponding finite-dimensional distributions of the MC $\{X(n)\}$. Thus we can assume without loss of generality that $f(y,z) = f(y_0, z)$ for arbitrary $y \in V$, $z \in \mathbf{Y}$. This yields, in particular, the equality $\tau_V(y) = \tau_V(y_0)$ for $y \in V$.

Applying Theorem 1, construct a stationary MC $\{X^n\}$ (also an "expanded" one) on a common probability space with $\{X(n)\}$ such that $\mu_0 < \infty$ a.s., where, as above,

$$\mu_0 = \min\{n \geq 1 : X(k) = X^k \text{ for all } k \geq n\}.$$

Here $\{X^n\}$ is also a SRS of the form $X^{n+1} = f(X^n, \xi_n)$. Let us consider, as in the previous section, the sequences $\{X_k(n) = U^{-k} X(n+k)\}$, and introduce random variables μ_k, $\mu^0 = \sup_{k \geq 0} \mu_k$.

Denote $\tau_k = \tau_k(x) = \min\{n \geq -k : X_k(n) \in V\}$ and $\tau^0 = \tau^0(x) = \sup_{k \geq 0} \tau_k(x)$. Note that the sequence $\{\tau_k + k; k \geq 0\}$ consists of identically distributed random variables. Thus

$$\mathbf{P}(\tau^0 \geq N) \leq \sum_k \mathbf{P}(\tau_k \geq N) = \sum_k \mathbf{P}(\tau_0 \geq N - k) \leq \mathbf{E}(\tau_0; \tau_0 \geq N),$$

and the right-hand side goes to zero as $N \to \infty$.

Introduce, for $k \geq 0$, the random variables $t_{1,k} = \tau_k$ and, for $l \geq 1$, $t_{l+1,k} = \min\{n > t_{l,k} : X_k(n) \in V\} \equiv t_{l,k} + \tau_{l+1,k}$, where the random variables $\tau_{l,k}$ have for $l > 1$ the distribution of $\tau_V(y_0)$; $\gamma_k(x) \equiv \gamma_k = \min_{l \geq 1}\{t_{l,k} : t_{l,k} \geq 0\}$ is the first non-negative hitting time of V by $\{X_k(n)\}$; $\gamma = \sup_{k \geq 0} \gamma_k \equiv \gamma(x)$.

Then $\mathbf{P}(\gamma > 2N) \leq \mathbf{P}(\tau^0 > N) + \mathbf{P}(\tau_0 \leq N, \gamma > 2N)$. Let us perform a shift U^{-N}:

$$\mathbf{P}(\tau^0 \leq N; \gamma > 2N) \leq \mathbf{P}(\gamma(y_0) > N) =$$

$$= \mathbf{P}\Big(\bigcup_{\kappa=0}^{\infty} \tau_k(y_0) > N\Big) \leq \mathbf{E}\{\tau_V(y_0); \tau_V(y_0) \geq N\},$$

and the right-hand side goes to zero as $N \to \infty$. Thus

$$\mathbf{P}(\mu^0 > M) \leq \mathbf{P}(\gamma > N) + \mathbf{P}(\gamma \leq N, \mu^0 > M) \leq$$

$$\leq \mathbf{P}(\gamma > N) + \sum_{k=0}^{N} \mathbf{P}(\tilde{\mu} > M - k),$$

where $\tilde{\mu} = \min\{n \geq 1 : X(y_0, n) = X^n\}$.

Since the second summand on the right-hand side of the latter inequality tends to zero as $M \to \infty$ for any fixed N, the a.s. finiteness of μ^0 is proved.

Let us now introduce the converse of the second part of Theorem 8 (its proof will be published in another paper).

Theorem 9. *Let a MC* $\{X(n)\}$ *of the form* (7) *for some* $X(0) = x \in \mathbf{X}$ *and a stationary MC* $\{X^n\}$ *be defined on same probability space. Assume that* $\{X(n)\} \xrightarrow{sc} \{X^n\}$. *Then there exist a subspace* $\mathbf{X}^1 \subseteq \mathbf{X}$, *a set* $V \subseteq \mathbf{X}^1$, *numbers* p, m, *and a probability measure* φ *on* $\mathbf{X}^1$ *such that Conditions* (I)-(III) *of Theorem* 1 *are satisfied on* $\mathbf{X}^1$.

Moreover, it is possible to define a "positive" stationary renovating sequence $\{A_n\}$, $A_n \in \mathbf{F}^{\xi}_{n+m}$, *of the form* $A_n = A_n^{(1)} \cap A_n^{(2)}$, *where* $A_n^{(1)} \in \mathbf{F}^{\xi}_{n-1}$, $A_n^{(2)} \in \mathbf{F}^{\xi}_{n,n+m}$, *and a number* $M \geq 0$ *such that* $\{X(n) \in V\} \supseteq A_n^{(1)}$ *for any* $n \geq M$, *and the sequence* $G_n \equiv$ $\equiv \{X(n) \in V\} \cap A_n^{(2)} \supseteq A_n$ *is renovating for* $\{X(n)\}$.

If $\mu_1 = \min\{n \geq 0 : I(G_n) = 1\}$; $\mu_{k+1} = \min\{n > \mu_k + m : I(G_n) = 1\}$; $k \geq 1$ *denote the successive times of the realization of the events* G_n, *then the random variables* $\nu_k = \mu_{k+1} - \mu_k$, $k \geq 1$ *are i.i.d. (and do not depend on* μ_1*). Moreover,* $\mathbf{E}\,\nu_k < \infty$, *and the greatest common divisor of the set* $\{i : \mathbf{P}(\nu_k = i) > 0\}$ *equals* 1.

Let us comment on some difference between the statements of Theorems 2 and 9: in the conditions of Theorem 9 the convergence (sc-convergence) is required for some $x \in \mathbf{X}$ at least, while in the conditions of Theorem 2 the convergence (in total variation or coupling-convergence) is stipulated for all $x \in \mathbf{X}$.

Theorems 8 and 9 establish the interrelations between Conditions (I)-(III) and existence of "positive" stationary renovating events. However these interrelations may be observed directly after a thorough consideration of the ergodicity conditions in Theorem 1.

As noted above, the proof of Theorem 1 (see [7], [8]) is based on the fact, that Conditions (I)-(III) provide a possibility of expanding the phase space $\mathbf{X}$ to $\mathbf{X}^*$ and of constructing on the latter space a new chain $X^*(n) = (X(n), \delta_n)$ in such a way that $X(n)$ is its first coordinate and the new chain $\{X^*(n)\}$ obtains the regeneration property which makes it easy to prove its ergodicity. Then the chain $X^*(n)$ (or its phase space) can be further enlarged so that we obtain a chain $Y(n)$, which is equivalent to $X^*(n)$, and possesses a positive atom y_0 with the property $\{Y(n) = y_0\} \subseteq \{X(n) \in V\}$. Moreover, the time of recurrence to y_0 has finite expectation, and the greatest common divisor of the possible values of the recurrence time for non-periodic chains equals one (this implies immediately the ergodicity of $Y(n)$).

All this means that the event $A_n = \{Y(n) = y_0\}$ is renovating for $Y(n)$ and that $\mathbf{P}(A_n) > q > 0$ for all sufficiently large n. Therefore Conditions (I)-(III) mean that

1) there exists a sequence $A_n \in \mathbf{F}^*_n$ with $\mathbf{P}(A_n) \geq q > 0$ for sufficiently large n;

2) on the set A_n

$$\mathbf{P}(X^*(n+1) \in B \mid \mathbf{F}^*_n) = \varphi(B),$$

where the measure φ does not depend on $\mathbf{F}^*_n$ (here $\mathbf{F}^*_n = \sigma(X^*(0), \ldots, X^*(n))$, and for the sake of simplicity we assume $m = 0$).

On the other hand, the conditions of Theorem 3 may be rewritten in the form (as above, $m = 0$):

1a) there exists a stationary sequence $A_n \in \mathbf{F}_n$, $\mathbf{P}(A_n) > 0$;

2a) on the set A_n

$$\mathbf{P}(X(n+1) \in B \mid \mathbf{F}_n) = I(g(\xi_n) \in B) \equiv \varphi(\xi_n, B). \tag{8}$$

In this form Conditions 1), 2) (viz., Conditions (I), (II)) and Conditions 1a), 2a) (viz., the conditions of Theorem 3) have a lot in common.

This similarity becomes even more definite, if we consider the generalizations of SRS treated in the next chapter (the so-called recursive chains). For them the technique of the phase space expansion, mentioned above, also proves to be most useful.

R e m a r k 2 . Let us note, that there exists an essential difference between MC and SRS from the point of their behaviour with respect to Conditions (I)-(II). As follows from Theorem 8, MC X can be represented in the form of SRS $X(n+1) = f(X(n), \xi_n)$, $X(0) = const$, where $\{\xi_n\}$ is a sequence of i.i.d. random variables. If we suppose that the MC X satisfies Condition (II) for $V = \mathbf{X}$, $m = 0$, i.e.,

$$\mathbf{P}(X(n+1) \in B \mid X(n)) \geq p \cdot \varphi(B) \text{ a.s.}, \tag{9}$$

then this chain satisfies Conditions (I)-(III) and, by Theorem 1, MC X is ergodic. Condition (9) coincides with following one:

$$\mathbf{P}(X(n+1) \in B \mid X(n), \xi_{n-1}, \xi_{n-2}, \dots) \geq p \cdot \varphi(B) \text{ a.s.} \tag{10}$$

Let now X be a SRS defined by the relations $X(n+1) = f(X(n), \xi_n)$, $X(0) = const$, where the driver $\{\xi_n\}$ is stationary and metrically transitive. Does (10) imply ergodicity of X in this case? The answer to this question is, in general, in the negative, as demonstrated by the following example.

E x a m p l e 5 . Let $\{\eta_n\}$ be a sequence of i.i.d. random variables with uniform distribution on $[-1, 1]$, and $\xi_n = (\eta_n, \eta_{n+1})$, so that the sequence ξ_n is metrically transitive. Introduce the function

$$g(x) = \max(0, \min(1, 1-x)) = \begin{cases} 0 & \text{for } x \geq 1, \\ 1-x & \text{for } 0 \leq x \leq 1, \\ 1 & \text{for } x < 0. \end{cases}$$

Define a real-valued SRS $X = \{X(n); n \geq 0\}$ in the following way:

$$X(0) = const \;,\; X(n+1) = g(X(n) - \eta_n) + \eta_{n+1} \equiv f(X(n), \xi_n).$$

Since $X(n)$ and η_n are measurable with respect to the σ-algebra generated by the random variables $X(n), \xi_{n-1}, \xi_{n-2}, \dots, \xi_0$, Condition (10) is satisfied for X, if one assumes $p = 1/2$ and defines φ as the Lebesgue measure on $[0, 1]$. On the other hand, note that for $n \geq 1$

$$g^{(2n-1)}(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 0 & \text{for } x \geq 1, \\ 1-x & \text{for } 0 < x < 1, \end{cases} \qquad g^{(2n)}(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x \geq 1, \\ x & \text{for } 0 < x < 1 \end{cases}$$

(here $g^{(i)}$ is the i -th iteration of g). Set $Y(n) = X(n) - \eta_n$, then

$$Y(n) = g(Y(n-1)) = \begin{cases} g(Y(1)) & \text{for even } n, \\ 1 - g(Y(1)) & \text{for odd } n. \end{cases}$$

But for $X(0) \neq 1/2$ the distributions of $Y^* = g(Y(1)) = g^{(2)}(X(0) - \eta_0)$ and $1 - g(Y(1)) = 1 - Y^*$ are different. Also different (and otherwise independent of n) are the distributions of $X(2n+1) = 1 - Y^* + \eta_{2n+1}$ and $X(2n) = Y^* + \eta_{2n}$. This means that although Condition (10) is satisfied, the distribution of SRS $\{X(n)\}$ is periodical (one may set $V = \mathbf{X} = [-2, 2]$). In other words, the transfer of Conditions (II) from MC to SRS in form (10) does not lead, in general, to the ergodicity of SRS. Thus to formulate an analog of Conditions (I)-(III) for SRS (and for recursive chains introduced in the next chapter), one faces the necessity to expand the σ-algebra, with respect to which the conditional expectation is taken, i.e., to consider a condition of type (8).

5. Estimates of the convergence rate

Let us now present some simple results concerning the estimates of the convergence rate. For a MC the rate of convergence to the stationary distribution can be estimated in terms of the distribution $\mathbf{P}(\tau_V(x) > n)$, $x \in V$, where $\tau_V(x) = \min\{n \geq 1 : X(x, n) \in V\}$ (see [6], [12]). In other words, the estimation is carried out using the distribution of lapses between successful hitting times of V for the sequence X.

For a SRS the estimates of the convergence rate are obtained similarly, using the hitting times of the sets A. More accurately, let $\{A_n\}$, $A_n \in \mathbf{F}^{\xi}_{n+m}$, be a stationary sequence of renovating events for the SRS $\{X(n)\}$ and τ the lapse between two successive realizations of A_n:

$$\mathbf{P}(\tau = k) = \mathbf{P}(A_k \cap \overline{A}_{k-1} \cap \ldots \cap \overline{A}_1 \mid A_0).$$

Then we can estimate the rate of convergence in terms of the distribution of τ (see [17]).

Theorem 10. *The following inequality is valid*:

$$\mathbf{P}(\mu^0 > n) \leq \mathbf{P}(A_0)\,\mathbf{E}(\tau - n + m)^+ .$$

Proof. Indeed,

$$\mathbf{P}(\mu^0 > n) \leq \mathbf{P}(\overline{A}_0 \cap \ldots \cap \overline{A}_{n-m-1}) = \mathbf{P}(\overline{A}_1 \cap \ldots \cap \overline{A}_{n-m}) =$$

$$= \mathbf{P}(A_0 \cap \overline{A}_1 \cap \ldots \cap \overline{A}_{n-m}) + \mathbf{P}(\overline{A}_0 \cap \overline{A}_1 \cap \ldots \cap \overline{A}_{n-m}) = \ldots =$$

$$= \sum_{i=n-m}^{\infty} \mathbf{P}(A_0)\,\mathbf{P}(\overline{A}_1 \cap \ldots \cap \overline{A}_i \mid A_0) = \mathbf{P}(A_0)\mathbf{E}(\tau - n + m)^+ ,$$

Q.E.D.

The estimates for a SRS with a non-stationary driver can be obtained similarly. For instance, let the conditions of Theorem 7 be satisfied. Denote, as above, by

$$\mu^0 = \min\{n \geq 0 : U^{-l} X(i + l) = X^i \text{ for all } l \geq 0 ; i \geq n\}$$

the "sc-time" of the sequences $\{X(k)\}$ and $\{X^k\}$, and by γ^0 the "sc-time" of the sequences $\{Z(n)\}$ and $\{Z^n\}$. Then

$$\mathbf{P}(\gamma^0 > 2n + M) \le \sum_{i=0}^{2n} \mathbf{P}(\mu^0 = i; \overline{D}_{i+1} \cap \ldots \cap \overline{D}_{2n}) +$$

$$+ \mathbf{P}(\mu^0 > 2n) \le \mathbf{P}(\overline{D}_1 \cap \ldots \cap \overline{D}_n) + \mathbf{P}(\mu^0 > n) =$$

$$= \mathbf{P}(\mu^0 > n) + \mathbf{P}(D_0) \cdot \mathbf{E}(T - n)^+ ,$$

where $\mathbf{P}(T = k) = \mathbf{P}(D_k \cap \overline{D}_{k-1} \cap \ldots \cap \overline{D}_1 \mid D_0)$.

CHAPTER 3. RECURSIVE CHAINS (MARKOV CHAINS IN RANDOM ENVIRONMENT)

1. Main definitions and properties

Assume as before that $\{X(n)\}_{n=0}^{\infty}$, $\{\xi_n\}_{n=0}^{\infty}$ are two random sequences with components $X(n)$ and ξ_n taking values in arbitrary measurable phase spaces $(\mathbf{X}, \mathbf{B_X})$ and $(\mathbf{Y}, \mathbf{B_Y})$ respectively. As before, we shall assume, if necessary that the σ-algebras $\mathbf{B_X}$ and $\mathbf{B_Y}$ are countably generated.

The sequence $\{\xi_n\}$ will be considered as specified in advance. The sequences $\{\xi_n\}$ and $\{X(n)\}$ are assumed to be defined on the same probability space. Denote by $\mathbf{F}_n$ the σ-algebra $\mathbf{F}_n = \sigma(\xi_0, \ldots, \xi_n; X(0), \ldots, X(n))$, generated by the "history" of the sequence $\{\xi_k, X(k)\}$ up to the time n .

Definition 1. We shall say that a sequence $\{X(n)\}$ is a *recursive chain (RC) with driver* $\{\xi_n\}$ if the equality

$$\mathbf{P}(X(n+1) \in B \mid \mathbf{F}_n) = \mathbf{P}(X(n+1) \in B \mid X(n), \xi_n) \text{ a.s.} \tag{1}$$

holds for all $n \ge 0$, $B \in \mathbf{B_X}$.

It follows from the fact that the σ-algebras are countably generated that there exist regular conditional probabilities in (1), viz., such functions $\mathbf{P}_{(n)} : \mathbf{X} \times \mathbf{Y} \times \mathbf{B_X} \to [0,1]$ that

1) $\mathbf{P}_{(n)}(x, y, \cdot)$ is a probability measure on $(\mathbf{X}, \mathbf{B_X})$ for all fixed x, y ;

2) the function $\mathbf{P}_{(n)}(x, y, B)$ is measurable with respect to the pair of variables (x, y) for each $B \in \mathbf{B_X}$;

3) the conditional probabilities satisfy the relations

$$\mathbf{P}(X(n+1) \in B \mid \mathbf{F}_n) = \mathbf{P}_{(n)}(X(n), \xi_n, B) \text{ a.s.} \tag{2}$$

or, equivalently,

$$\mathbf{P}(X(n+1) \in B \mid X(n), \xi_n) = \mathbf{P}_{(n)}(X(n), \xi_n, B) \text{ a.s.}$$

E x a m p l e 1 . To illustrate the intrinsic character of the introduced notion, let us consider as an example one of the many applied problems in which the random process characterizing the behaviour of the system parameters of interest is a recursive chain.

Consider a communication network which is a "random multiple access broadcast channel" [19]; such systems are treated in a fairly extensive collection of works (see, e.g., [20] for more detailed references). Let there be given a transmission channel of the messages (packages) connecting many users. The arrival times are discrete (integer-valued), the transmission time is equal to one. Each message is transmitted from one of the users to all the others. If two or more users transmit simultaneously, then the channel is blocked. The fact of blocking becomes known to the user, and the untransmitted message is subject to transmission. All the users apply the same protocol for the repeated transmission (retranslation) of the messages. It is chosen whenever possible to be autonomous, using no information about the number of other users' untransmitted messages. The protocol ALOHA prescribes to each untranslated message at any time $n = 1, 2, \ldots$ to claim repeated retranslation with probability p and "stand still" with probability $1 - p$.

Let us proceed to a more formal description of the system. Let ξ_n be the number of "new messages", which are presented for translation at time n, $X(n)$ the number of untranslated messages up to time n, and η_n the number of claims in the channel at time n generated by untransmitted messages.

We shall assume that the probabilities

$$\mathbf{P}(\eta_n = j \mid \xi_0, \ldots, \xi_n; X(0), \ldots, X(n)) = \mathbf{P}(\eta_n = j \mid X(n)) = q_{j, X(n)} ,$$

where $\sum_{j=0}^{N} q_{j,N} = 1$ for any $N = 0, 1, \ldots$, are specified. In particular, for the ALOHA protocol $q_{j,N} = C_N^j p^j (1 - p)^{N-j}$.

It follows from the above argument that

$$X(n+1) = \begin{cases} X(n) + \xi_n & \text{on the set } \{\xi_n + \eta_n \neq 1\}, \\ X(n) & \text{on the set } \{\xi_n = 1; \eta_n = 0\}, \\ X(n) - 1 & \text{on the set } \{\xi_n = 0; \eta_n = 1\}. \end{cases}$$

This enables us to write down the distribution of $X(n+1)$ for known $X(n)$ and ξ_n , which means that the sequence $X(n)$ is a RC with driver $\{\xi_n\}$.

Similarly to the function $\mathbf{P}(x, B)$ for MC, the functions $\mathbf{P}_{(n)}(x, y, B)$ could be named transition probabilities (or functions): they define the probability for the RC $\{X(n)\}$ to hit the set B at the $(n+1)$-th step starting from x, when the driver takes on the value y . If we fix all the trajectory of $\{\xi_n\}$, then the RC $\{X(n)\}$ may be treated as a non-homogeneous Markov chain with the transition function $\mathbf{P}_{(n)}(x, B) =$ $= \mathbf{P}_{(n)}(x, \xi_n, B)$ (evidently, this chain would be generally non-homogeneous, even in the case when $\mathbf{P}_{(n)}(x, y, B)$ do not depend on n, and the sequence $\{\xi_n\}$ is stationary, $\xi_n \not\equiv const$; see Chapter 4. All this means that the a RC $\{X(n)\}$ may be treated also as a *MC in "random environment"*, when the transition probabilities $\mathbf{P}_{(n)}(x, B)$ are chosen at random (from a certain family).

As mentioned above, for a MC the knowledge of the transition probabilities $\mathbf{P}(x, B)$ enables one to construct for a given initial value the unique Markov chain according to the measure extension theorem . We have a similar situation with RC: it is possible to construct the distribution of the sequence $\{X(n), \xi_n\}$ from the distribution of the sequence $\{\xi_n\}$ and from the collection of functions $\{\mathbf{P}_{(n)}\}$.

Indeed, let us assume for the sake of simplicity that the initial value $X(0) = x_0$ is constant (this does not lead to any loss of generality). The distribution of the pair $Z_0 = (\xi_0, X(1))$ will be defined by relation

$$\mathbf{P}(\xi_0 \in A_0 ; X(1) \in B_1) = \int \mathbf{P}(\xi_0 \in dy_0) \cdot \mathbf{P}_{(0)}(x_0, y_0, B_1). \tag{3}$$

Using the distribution of Z_{n-1}, that of $Z_n = (\xi_0, \dots, \xi_n ; X(0), \dots ; X(n+1))$ for $n > 0$ will be defined by the relation

$$\mathbf{P}\Big(\bigcap_{k=0}^{n} \{\xi_k \in A_k ; X(k+1) \in B_{k+1}\}\Big) =$$

$$= \int \dots \int_{E_{n-1}} \mathbf{P}\Big(\bigcap_{k=0}^{n-1} \{\xi_k \in dy_k ; X(k+1) \in dx_{k+1}\}\Big) \times$$

$$\mathbf{P}\Big(\xi_n \in dy_n \mid \bigcap_{k=0}^{n-1} \{\xi_k \in dy_k\}\Big) \cdot \mathbf{P}_{(n)}(x_n, y_n, B_{n+1}),$$

where E_{n-1} is the set $\{y_0 \in A_0, \dots, y_{n-1} \in A_{n-1} ; x_1 \in B_1, \dots, x_n \in B_n\}$.

These relations can also be rewritten in the form

$$\mathbf{P}(\xi_n \in A_n, X(n+1) \in B_{n+1} \mid \sigma(Z_{n-1})) =$$

$$= \int \mathbf{P}(\xi_n \in dy_n \mid \xi_0, \dots, \xi_{n-1}) \cdot \mathbf{P}_{(n)}(X(n), y_n, B_{n+1}). \tag{4}$$

Assuming $B_{n+1} = \mathbf{X}$, we obtain

$$\mathbf{P}(\xi_n \in A_n \mid \sigma(Z_{n-1})) = \mathbf{P}(\xi_n \in A_n \mid \xi_0, \dots, \xi_{n-1}) \tag{5}$$

(this is the property of conditional independence of ξ_n from $X(n), \dots, X(0)$ for given $\mathbf{F}^{\xi}_{n-1}$).

Let us demonstrate that the distributions of Z_n so constructed or, equivalently, the finite-dimensional distributions of $\{\xi_k, X(k)\}$, possess the recursive property of (2). Indeed, by (5) relation (4) can be rewritten in the form

$$\mathbf{P}(\xi_n \in A_n ; X(n+1) \in B_{n+1} \mid \sigma(Z_{n-1})) =$$

$$= \int \mathbf{P}(\xi_n \in dy_n \mid \sigma(Z_{n-1})) \cdot \mathbf{P}_{(n)}(X(n), y_n, B_{n+1}) =$$

$$= \mathbf{E}(\mathbf{P}_{(n)}(X(n), \xi_n, B_{n+1}) ; \xi_n \in A_n \mid \sigma(Z_{n-1})),$$

which is, evidently, equivalent to

$$\mathbf{P}(X(n+1) \in B_{n+1} \mid \sigma(Z_{n-1}, \xi_n)) \equiv \mathbf{P}(X(n+1) \in B_{n+1} \mid \mathbf{F}_n) =$$

$$= \mathbf{P}_{(n)}(X(n), \xi_n, B_{n+1})$$

(recall that $\sigma(Z_{n-1}, \xi_n) = \mathbf{F}_n$). This proves (2).

Thus, the knowledge of the distribution of $\{\xi_n\}$ and the functions $\mathbf{P}_{(n)}(x, y, B)$ enables one to determine the finite-dimensional distributions of the sequence $Z = \{(X(n), \xi_n)\}$ so that they satisfy (2). Hence, according to the measure extension theorem, there exists, in the space $(\mathbf{Z}, \mathbf{B_Z}) = ((\mathbf{X} \times \mathbf{Y})^{\infty}, (\mathbf{B_X} \times \mathbf{B_Y})^{\infty})$, the distribution of this sequence such that (2) holds.

The question, whether the distribution of Z in $(\mathbf{Z}, \mathbf{B_Z})$ satisfying (2) for given functions $\mathbf{P}_{(n)}$ is unique, is still open in general. However, if we require in addition that the distribution of Z satisfying (2) possess also the property (5) (viz., that of ξ_n being conditionally independent of $X(n), \dots, X(0)$), then such a distribution in $(\mathbf{Z}, \mathbf{B}_{\mathbf{Z}})$ will be unique. In other words, the finite-dimensional distributions of the sequence Z will be uniquely defined by formulae (4). Indeed, the necessity of defining the distribution of $Z_0 = (\xi_0, X(1))$ by relation (3) is evident, since due to the conditions above

$$\mathbf{P}(X(1) \in B_1 \mid \xi_0, X(0)) = \mathbf{P}_{(0)}(x_0, \xi_0, B_1),$$

and thus

$$\mathbf{P}(\xi_0 \in A_0, X(1) \in B_1) = \mathbf{E}\{\mathbf{P}(X(1) \in B_1 \mid \xi_0, X(0)); \xi_0 \in A_0\} =$$

$$= \mathbf{E}\{\mathbf{P}_{(0)}(x_0, \xi_0, B_1); \xi_0 \in A_0\} = \int_{A_0} \mathbf{P}(\xi_0 \in dy_0)\, \mathbf{P}_{(0)}(x_0, y_0, B_1).$$

For the sake of brevity let us denote by C_n the direct product $\prod_{k=0}^{n} (A_k \times B_{k+1})$. This is a cylindrical set in the phase space of random variables Z_n. Then

$$\mathbf{P}(Z_n \in C_n) = \mathbf{E}[\mathbf{P}(\xi_n \in A_n; X(n+1) \in B_{n+1} \mid \sigma(Z_{n-1})); Z_{n-1} \in C_{n-1}] =$$

$$= \mathbf{E}[\mathbf{E}\{I(\xi_n \in A_n) \cdot \mathbf{P}(X(n+1) \in B_{n+1} \mid \mathbf{F}_n) \mid \sigma(Z_{n-1})\}; Z_{n-1} \in C_{n-1}] =$$

$$= \mathbf{E}[\mathbf{E}\{I(\xi_n \in A_n) \cdot \mathbf{P}_{(n)}(X(n), \xi_n, B_{n+1}) \mid \sigma(Z_{n-1})\}; Z_{n-1} \in C_{n-1}]. \tag{6}$$

But by virtue of (5) (recall that $X(n)$ is measurable with respect to $\sigma(Z_{n-1})$)

$$\mathbf{E}\{I(\xi_n \in A_n) \cdot \mathbf{P}_{(n)}(X(n), \xi_n, B_{n+1}) \mid \sigma(Z_{n-1})\} =$$

$$= \int_{A_n} \mathbf{P}(\xi_n \in dy_n \mid \xi_0, \dots, \xi_{n-1}) \cdot \mathbf{P}_{(n)}(X(n), y_n, B_{n+1}).$$

If we substitute the latter into equalities (6) and compare the beginning and the end of those, we obtain (4), Q.E.D.

Let us make a general remark. In applications (see, for instance, Example 1) the sequence $\{\xi_n\}$ is always *specified in advance*. It is defined by some external factors, and relation (1) appears usually as one *characterizing* the sequence $\{X(n)\}$, i.e., enabling the sequential construction of the latter from the elements ξ_n (for instance, by means of the most natural procedure introduced above). These circumstances enable us to postulate along with (1) another property:

The distribution of ξ_n *for given* $\xi_0, \dots, \xi_{n-1}$ *does not depend on* $X(0), \dots, X(n)$.

But this is Condition (5).

So *here and in the sequel we shall consider* (5) *to be justified and therefore assume that the distribution of* Z *in* $(\mathbf{Z}, \mathbf{B}_{\mathbf{Z}})$, *satisfying* (2), *is unique.* It means that formulae (4), which specify the finite-dimensional distributions of Z, are valid.

It is clear that a RC is a more general object than a MC: any MC is by definition a RC for the trivial driver $\xi_n \equiv$ *const*.

If random variables ξ_n are independent, then a RC also forms a MC. Indeed,

$$\mathbf{P}(X(n+1) \in B_{n+1} \mid X(0), \dots, X(n)) =$$

$$= \mathbf{E}[\mathbf{P}(X(n+1) \in B_{n+1} \mid \mathbf{F}_n) \mid X(0), \dots, X(n)] =$$

$$= \mathbf{E}[\mathbf{P}_{(n)}(X(n), \xi_n, B_{n+1}) \mid X(0), \dots, X(n)] =$$

$$= \mathbf{E}\{\mathbf{E}[\mathbf{P}_{(n)}(X(n), \xi_n, B_{n+1}) \mid \sigma(Z_{n-1})] \mid X(0), \dots, X(n)\}.$$

Since (5) is valid and $X(n)$ is measurable with respect to $\sigma(Z_{n-1})$, the relations

$$\mathbf{E}[\mathbf{P}_{(n)}(X(n), \xi_n, B_{n+1}) \mid \sigma(Z_{n-1})] = \int \mathbf{P}(\xi_n \in dy_n) \cdot \mathbf{P}_{(n)}(X(n), y_n, B_{n+1})$$

become valid. Thus

$$\mathbf{P}(X(n+1) \in B_{n+1} \mid X(0), \dots, X(n)) = \int \mathbf{P}(\xi_n \in dy_n) \cdot \mathbf{P}_{(n)}(X(n), y_n, B_{n+1})$$

and the left-hand side depends on $X(n)$ only. It stipulates the Markov property of $\{X(n)\}$.

Theorem 1. *Conditions* (2) *and* (5) *are jointly equivalent to the following one:*

$$\mathbf{P}(X(n+1) \in B \mid X(0), \dots, X(n); \{\xi_k\}_{k=0}^{\infty}) = \mathbf{P}(X(n+1) \in B \mid X(n), \xi_n) \quad (7)$$

a.s. for any $n \geq 0$ *and* $B \in \mathbf{B}_{\mathbf{X}}$.

Proof. Let (2) and (5) be satisfied. For arbitrary $n \geq 0$ we shall use abbreviation

$$D_n \equiv D_n(x_1, \dots, x_n; y_0, \dots, y_n) =$$

$$= \{X(k) \in dx_k; \xi_k \in dy_k; k = 1, \dots, n; \xi_0 \in dy_0\}.$$

For any $B \in \mathbf{B}_{\mathbf{X}}$; $H_1, \dots, H_k \in \mathbf{B}_{\mathbf{Y}}$ the equalities

$$\mathbf{P}(X(n+1) \in B; \xi_{n+1} \in H_1, \dots, \xi_{n+k} \in H_k \mid D_n) =$$

$$= \int_B \mathbf{P}(X(n+1) \in dx_{n+1} \mid D_n) \cdot \int_{H_1} \mathbf{P}(\xi_{n+1} \in dy_{n+1} \mid D_n; X(n+1) \in dx_{n+1}) \cdot$$

$$\cdot \ \dots \cdot \int_{H_k} \mathbf{P}(\xi_{n+k} \in dy_{n+k} \mid D_n; X(n+1) \in dx_{n+1}; \xi_{n+1} \in dy_{n+1}; \dots; \xi_{n+k-1} \in dy_{n+k-1})$$

hold a.e. in y_i with respect to the distribution of ξ_i; $i = 0, \dots, n$, and a.e. in x_i with respect to the distribution of $X(i)$; $i = 1, \dots, n$.

By Condition (5) one obtains the relation

$$\mathbf{P}(\xi_{n+j} \in dy_{n+j} \mid D_n; X(n+1) \in dx_{n+1}; \xi_{n+1} \in dy_{n+1}; \dots; \xi_{n+j-1} \in dy_{n+j-1}) =$$

$$= \mathbf{P}(\xi_{n+j} \in dy_{n+j} \mid \xi_0 \in dy_0; \dots; \xi_{n+j-1} \in dy_{n+j-1})$$

a.e. in y_i with respect to the distribution of ξ_i; $i = 0, 1, \dots, n+j-1$, and a.e. in x_i with respect to the distribution of $X(i)$; $i = 1, \dots, n+1$. Due to (2)

$$\mathbf{P}(X(n+1) \in dx_{n+1} \mid D_n) = \mathbf{P}(X(n+1) \in dx_{n+1} \mid X(n) \in dx_n; \xi_n \in dy_n)$$

a.e. in y_i with respect to the distribution of ξ_i; $i = 0, \dots, n$ and a.e. in x_i with respect to the distribution of $X(i)$, $i = 1, \dots, n$. Then the equality

$$\mathbf{P}(X(n+1) \in B; \xi_{n+1} \in H_1; \dots; \xi_{n+k} \in H_k \mid D_n) =$$

$$= \mathbf{P}_{(n)}(x_n, y_n, B) \cdot \mathbf{P}(\xi_{n+1} \in H_{n+1}; \ldots; \xi_{n+k} \in H_{n+k} \mid \xi_0 \in dy_0, \ldots, \xi_n \in dy_n)$$

holds for these (x_i, y_i). Hence,

$$\mathbf{P}(X(n+1) \in B \mid X(0), \ldots, X(n); \xi_0, \ldots, \xi_{n+k}) = \mathbf{P}_{(n)}(X(n), \xi_n) \text{ a.s.}$$

The choice of $k \geq 0$ being arbitrary, we obtain (7).

Conversely, let (7) be satisfied. Let us demonstrate the validity of (5). Indeed, the following equalities

$$\mathbf{P}(X(1) \in B_1, \ldots, X(n+1) \in B_{n+1}; \xi_{n+1} \in A \mid \xi_0 \in dy_0, \ldots, \xi_n \in dy_n) =$$

$$= \int_A \mathbf{P}(\xi_{n+1} \in dy_{n+1} \mid \xi_0 \in dy_0, \ldots, \xi_n \in dy_n) \cdot \int_{B_1} \mathbf{P}(X(1) \in dx_1 \mid$$

$$\xi_0 \in dy_0, \ldots, \xi_{n+1} \in dy_{n+1}) \cdot \ldots \cdot \int_{B_{n+1}} \mathbf{P}(X(n+1) \in dx_{n+1} \mid \xi_0 \in dy_0,$$

$$\ldots, \xi_{n+1} \in dy_{n+1}; X(1) \in dx_1, \ldots, X(n) \in dx_n) =$$

$$= \int_A \mathbf{P}(\xi_{n+1} \in dy_{n+1} \mid \xi_0 \in dy_0, \ldots, \xi_n \in dy_n) \cdot \int_{B_1} \mathbf{P}(X(1) \in dx_1 \mid \xi_0 \in dy_0) \cdot$$

$$\cdot \ldots \cdot \int_{B_{n+1}} \mathbf{P}(X(n+1) \in dx_{n+1} \mid \xi_n \in dy_n; X(n) \in dx_n) =$$

$$= \mathbf{P}(\xi_{n+1} \in A \mid \xi_0 \in dy_0, \ldots, \xi_n \in dy_n) \cdot \mathbf{P}(X(1) \in B_1, \ldots,$$

$$X(n+1) \in B_{n+1} \mid \xi_0 \in dy_0, \ldots, \xi_n \in dy_n)$$

take place for any $n \geq 0$, all sets $A \in \mathbf{B_Y}$, $B_i \in \mathbf{B_X}$, $i = 1, \ldots, n+1$, and a.e. in y_i with respect to the distribution of ξ_i ; $i = 0, \ldots, n$. This leads us to (5). Theorem 1 is proved.

Let now $\{\xi_n\}$, $\{\eta_n\}$ be two random sequences with elements ξ_n, η_n assuming values in arbitrary measurable phase spaces $(\mathbf{Y}, \mathbf{B_Y})$ and $(\mathbf{Z}, \mathbf{B_Z})$ respectively. Suppose that $\{X(n)\}$ is a RC with values in $(\mathbf{X}, \mathbf{B_X})$ and driver $\{(\xi_n, \eta_n)\}$: for $B \in \mathbf{B_X}$

$$\mathbf{P}(X(n+1) \in B \mid X(0), \ldots, X(n); \{\xi_i, \eta_j\}) = \tilde{\mathbf{P}}_{(n)}(X(n), \xi_n, \eta_n, B) \text{ a.s.} \tag{8}$$

As before, we shall assume the σ-algebras $\mathbf{B_X}, \mathbf{B_Y}, \mathbf{B_Z}$ to be countably generated. Denote for $n \geq 0$, $x \in \mathbf{X}$, $y \in \mathbf{Y}$, $B \in \mathbf{B_X}$

$$\mathbf{P}_{(n)}(x, y, B) = \mathbf{E}\tilde{\mathbf{P}}_{(n)}(x, y, \eta_n, B). \tag{9}$$

We shall utilize the following theorem.

Theorem 2. *Consider a RC* $\{X(n)\}$ *with driver* $\{(\xi_n, \eta_n)\}$ *and transition functions* $\tilde{\mathbf{P}}_{(n)}(x, y, z, B)$ *such that* $\{X(n)\}$ *satisfies* (8). *Suppose that the elements of the sequence* $\{\eta_n\}$ *are i.i.d., and* $\{\eta_n\}$ *does not depend on* $\{\xi_n\}$. *Then* $\{X(n)\}$ *is a RC with driver* $\{\xi_n\}$ *and transition functions* $\mathbf{P}_{(n)}(x, y, B)$ *of the form* (9) *and* $\{X(n)\}$ *satisfies* (5) (*or, equivalently,* (7)).

Proof. The conditions of the theorem imply necessarily that for each n the value η_n does not depend on the totality of random variables $\{ X(i), i \le n; \xi_j, -\infty < j < \infty \}$. Thus the equality

$$\mathbf{P}(X(n+1) \in B \mid \mathbf{F}_n^{(1)}) = \mathbf{E}\{ \mathbf{E}\{ I(X(n+1) \in B) \mid \mathbf{F}_n^{(2)} \} \mid \mathbf{F}_n^{(1)} \} =$$

$$= \mathbf{E}\{ \tilde{\mathbf{P}}_{(n)}(X(n), \xi_n, \eta_n, B) \mid \mathbf{F}_n^{(1)} \} = \mathbf{P}_{(n)}(X(n), \xi_n, B) \text{ a.s.}$$

holds for each set $B \in \mathbf{B}_{\mathbf{X}}$ and $n \ge 0$, where $\mathbf{F}_n^{(1)} = \sigma\{ X(i); i \le n; \{\xi_j\} \}$ and $\mathbf{F}_n^{(2)} =$ $= \sigma\{ X(i); i \le n; \{ \xi_j, \eta_j \} \}$. Hence, Condition (7) is satisfied. Theorem 2 is proved.

2. Reduction of RC to SRS

Let, as before, $(\mathbf{X}, \mathbf{B_X})$ and $(\mathbf{Y}, \mathbf{B_Y})$ be arbitrary phase spaces, $\{\xi_n\}$ a given random sequence assuming values in $(\mathbf{Y}, \mathbf{B_Y})$. In this section we shall assume that $\mathbf{B}_{\mathbf{X}}$ and $\mathbf{B}_{\mathbf{Y}}$ are countably generated.

Along with SRS introduced in Chapter 2, one may consider the more general concept of a "non-homogeneous" stochastically recursive sequence, with the stationarity of $\{\xi_n\}$ not assumed and the functions f dependent on n.

So let a collection $f_{(n)} : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$, $n \ge 0$ of measurable functions be given.

Definition 2. We shall say that $\{X(n)\}$ is a *SRS with driver* $(\{\xi_n\}, \{f_{(n)}\})$ if

$$X(n+1) = f_{(n)}(X(n), \xi_n) \quad n \ge 0 \text{ a.s.}$$

If all the functions $f_{(n)} \equiv f$ coincide and the sequence $\{\xi_n\}$ is stationary, it is natural to call such SRS a *homogeneous* one; thus in Chapter 2 we have studied *homogeneous SRS.*

It is clear that the values of an arbitrary SRS $\{X(n)\}$ are connected into a recursive chain. Indeed,

$$\mathbf{P}(X(n+1) \in B_{n+1} \mid \mathbf{F}_n) = \mathbf{P}(f_{(n)}(X(n), \xi_n) \in B_{n+1} \mid \mathbf{F}_n) \equiv$$

$$\equiv \mathbf{P}(f_{(n)}(X(n), \xi_n) \in B_{n+1} \mid X(n), \xi_n) = I(f_{(n)}(X(n), \xi_n) \in B_{n+1}).$$

The converse is also true in a certain sense: any RC may be represented in the form of SRS (generally, with another driver). This fact facilitates and simplifies the study of RC.

Theorem 3. *Let the sequence* $\{X(n)\}$ *form a RC with driver* $\{\xi_n\}$ *and Conditions* (2), (5) *be satisfied. Then it is possible to define a collection of measurable functions* $f_{(n)} : \mathbf{X} \times \mathbf{Y} \times [0,1] \to \mathbf{X}$, *a sequence* $Z = \{(X(n), \xi_n)\}_{n=0}^{\infty}$, *and a sequence* $\{\alpha_n\}_{n=0}^{\infty}$ *of independent r.v.'s, uniformly distributed on the segment* $[0,1]$, *on a common probability space so that*

1) *the sequences* $\{\alpha_n\}$ *and* $\{\xi_n\}$ *are independent;*

2) $\{X(n)\}$ *forms a SRS with driver* $(\{\xi_n, \alpha_n\}, \{f_n\})$.

Proof. Let us start from the case when $\mathbf{X} = \mathbf{Y} = [0,1]$ and the σ-algebras $\mathbf{B}_{\mathbf{X}}$, $\mathbf{B}_{\mathbf{Y}}$ consist of Borel sets. Denote

$$F_{(n)}(t \mid X(n), \xi_n) = \mathbf{P}(X(n+1) < t \mid X(n), \xi_n) \equiv \mathbf{P}_{(n)}(X(n), \xi_n, (-\infty, t)).$$

Let us define the functions $f_{(n)}$ by equalities $f_{(n)}(x, y, t) = F_{(n)}^{-1}(t \mid x, y)$.

As elucidated above, without loss of generality we can consider RC $\{X(n)\}$ to be constructed by means of the procedure introduced in section 1 of this chapter, where the sequence $\{\xi_n\}$ is the initial material of the construction and is specified in advance. We shall reproduce the same procedure in a slightly different form.

Take a random variable α_0, uniformly distributed on $[0,1]$ and independent of $X(0)$ and $\{\xi_n\}$, and set $X(1) = f_{(0)}(X(0), \xi_0, \alpha_0)$.

Let us then apply an induction argument. Suppose that the random variables $X(0), \dots, X(n)$; $\alpha_0, \dots, \alpha_{n-1}$ are constructed for some $n \ge 1$. Define α_n as a random variable, uniformly distributed on $[0,1]$ and independent of all the random variables $X(0), \dots, X(n)$, $\alpha_0, \dots, \alpha_{n-1}$, $\{\xi_k\}_{k=0}^{\infty}$; and set $X(n+1) = f_{(n)}(X(n), \xi_n, \alpha_n)$.

It is easily seen that the sequence $\{X(n)\}$ thus constructed satisfies (2), (5). Indeed,

$$\mathbf{P}(X(n+1) \in B \mid X(n), \dots, X(0); \xi_n, \dots, \xi_0) =$$

$$= \mathbf{P}(f_{(n)}(X(n), \xi_n, \alpha_n) \in B \mid X(n), \xi_n) =$$

$$= \mathbf{P}(F_{(n)}^{-1}(\alpha_n \mid \xi_n; X(n)) \in B \mid \xi_n, X(n)) = \mathbf{P}_{(n)}(X(n), \xi_n, B).$$

Let us proceed to the general case. Let $B_1, B_2, \dots$ be the basis sets of the σ- algebra $\mathbf{B}_{\mathbf{X}}$, i.e., assume that $\mathbf{B}_{\mathbf{X}} = \sigma\{B_k; k \ge 1\}$. We introduce an equivalence relation for the points x of the space $\mathbf{X}$: $x_1 \sim x_2$ if for any $k \ge 1$ either $x_1 \in B_k$ and $x_2 \in B_k$ simultaneously, or $x_1 \notin B_k$ and $x_2 \notin B_k$. Let us denote by $\tilde{x}$ (with or without subscripts) the equivalence classes, and by $\tilde{\mathbf{X}} = \{\tilde{x}\}$ the set of the equivalence classes. Each point $\tilde{x} \in \tilde{\mathbf{X}}$ can be uniquely associated with a $\{0,1\}$-valued sequence $s(\tilde{x}) = (s_1(\tilde{x}), \dots, s_k(\tilde{x}), \dots)$, where $s_n(\tilde{x}) = 1$, if $x \in B_n$ for $x \in \tilde{x}$ and $s_n(\tilde{x}) = 0$ otherwise. Thus it is reasonable to identify $\tilde{x}$ with $s(\tilde{x})$ and to write down $\tilde{x}_n$ instead of $s_n(\tilde{x})$. Then the set $\tilde{\mathbf{X}}$ will be a subspace of the space of $\{0,1\}$ -valued sequences $\mathbf{S} = \{s = (s_1, s_2, \dots)\}$. If we introduce in $\mathbf{S}$ the cylindrical σ-algebra $\mathbf{B}_{\mathbf{S}}$, then $\tilde{\mathbf{X}}$ will be one of its elements, since $\tilde{\mathbf{X}} = \bigcap_{n=1}^{\infty} \tilde{\mathbf{X}}^n$, where $\tilde{\mathbf{X}}^n = \{ s = (s_1, s_2, \dots) :$ there exists a $\tilde{x} \in \tilde{\mathbf{X}}$ such that $s_1 = \tilde{x}_1, \dots, s_n = \tilde{x}_n \}$ is an element of $\mathbf{B}_{\mathbf{S}}$.

Consider the space $(\tilde{\mathbf{X}}, \mathbf{B}_{\tilde{\mathbf{X}}})$, where $\mathbf{B}_{\tilde{\mathbf{X}}} = \mathbf{B}_{\mathbf{S}} \cap \tilde{\mathbf{X}}$. There is an intrinsic one-to-one correspondence $Q : \mathbf{B}_{\mathbf{X}} \leftrightarrow \mathbf{B}_{\tilde{\mathbf{X}}}$ between the σ-algebras $\mathbf{B}_{\mathbf{X}}$ and $\mathbf{B}_{\tilde{\mathbf{X}}}$. Here each random variable $\eta : \langle \Omega, \mathbf{F}, \mathbf{P} \rangle \to (\mathbf{X}, \mathbf{B}_{\mathbf{X}})$ can be associated with a random variable $\tilde{\eta}$: $\langle \Omega, \mathbf{F}, \mathbf{P} \rangle \to (\tilde{\mathbf{X}}, \mathbf{B}_{\tilde{\mathbf{X}}})$ according to the rule

$$\{\eta \in B\} = \{\tilde{\eta} \in QB\}. \qquad (10)$$

Two random variables η_1, η_2 with values in $(\mathbf{X}, \mathbf{B}_{\mathbf{X}})$ will be said to be equivalent, $\eta_1 \sim \eta_2$, if $\mathbf{P}(\eta_1 \in B, \eta_2 \in B) = \mathbf{P}(\eta_1 \in B)$ for any set $B \in \mathbf{B}_{\mathbf{X}}$.

Two random variables $\tilde{\eta}_1, \tilde{\eta}_2$ with values in $(\tilde{\mathbf{X}}, \mathbf{B}_{\tilde{\mathbf{X}}})$ will be said to be equivalent, if $\mathbf{P}(\tilde{\eta}_1 \ne \tilde{\eta}_2) = 0$.

It is easy to see that η_1 and η_2 are equivalent if and only if $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are equivalent. Thus relation (10) defines a one-to-one correspondence to within the equivalence. The mapping $\eta \to \tilde{\eta}$ will also be denoted by the symbol Q. Moreover, if $\eta_1 \sim \eta_2$, then the equality

$$\mathbf{P}(\eta_1 \in B, \eta_3 \in C) = \mathbf{P}(\eta_2 \in B, \eta_3 \in C)$$

holds for any random variable $\eta_3 : <\Omega, \mathbf{F}, \mathbf{P}> \to (\mathbf{X}, \mathbf{B_X})$ and for any $B, C \in \mathbf{B_X}$.

In its turn, the space $(\tilde{\mathbf{X}}, \mathbf{B}_{\tilde{\mathbf{X}}})$ could be mapped into the segment $[0,1]$ according to the rule: $H(\tilde{x}) = \sum_{i=1}^{\infty} \tilde{x}_i \cdot 3^{-i}$. Denote the range of values of $H(\tilde{x})$ by $\tilde{\tilde{x}}$. The set $\tilde{\tilde{\mathbf{X}}}$ is a Borel one. Moreover, if we denote by $\mathbf{B}_{\tilde{\tilde{\mathbf{X}}}}$ the σ-algebra of Borel sets on $\tilde{\tilde{\mathbf{X}}}$, then the mapping $H : (\tilde{\mathbf{X}}, \mathbf{B}_{\tilde{\mathbf{X}}}) \to (\tilde{\tilde{\mathbf{X}}}, \mathbf{B}_{\tilde{\tilde{\mathbf{X}}}})$ is measurable and one-to-one (as well as H^{-1}).

Hence, if we introduce similarly $\tilde{\mathbf{Y}}$, $\tilde{\tilde{\mathbf{Y}}}$ and define the mappings Q', H', we can pass from the random variables $\{X(n)\}$ and $\{\xi_n\}$ to the variables $\{H((X(n)))\}$ and $\{H'(Q'(\xi_n))\}$ with values on $[0,1]$; define for these the random variables $\{\alpha_n\}$; and return to the initial characteristics by inverse transformations. The theorem is proved.

The proof of the theorem implies

Corollary 1. *If the functions* $\mathbf{P}_{(n)} \equiv \mathbf{P}$ *coincide for different* n, *then the functions* $f_{(n)} \equiv f$ *also coincide.*

Moreover, it is evident that if $\{\xi_n\}$ is a stationary metrically transitive sequence, the sequence $\{(\xi_n, \alpha_n)\}$ also possesses these properties.

As it was already mentioned in Chapter 1, it follows from Theorem 3 that both SRS and RC are objects more general than MC: if $\{X(n)\}$ is a MC, than $\{X(n)\}$ is a RC, i.e., Condition (1) is satisfied for $\xi_n \equiv const$ and thus, by virtue of Theorem 3, $\{X(n)\}$ is a SRS with driver $\{\alpha_n\}$. Note that this yields us one more useful characterization of MC.

On the other hand, it is evident that if $\{\xi_n\}$ are arbitrary independent random variables, the SRS $\{X(n)\}$ forms a MC.

Definition 3. Let us call a RC $\{X(n)\}$ homogeneous if

1) the sequence $\{\xi_n\}$ is stationary;

2) the functions $\mathbf{P}_{(n)}$ do not depend on n:

$$\mathbf{P}_{(n)}(x, y, B) \equiv \mathbf{P}(x, y, B) \text{ for all } n, y, x, B.$$

Corollary 1 and the above remarks imply

Corollary 2. *If* $\{X(n)\}$ *is a homogeneous RC with driver* $\{\xi_n\}$, *then it is a homogeneous SRS with driver* $\{(\xi_n, \alpha_n)\}$.

Note that for the case when $\{X(n)\}$ assume values on the real line (i.e., $\mathbf{X} = \mathbf{R}$), RC were introduced and constructively reduced to SRS in [21].

In the sequel we shall consider only homogeneous RC and SRS. Therefore we omit the word "homogeneous".

CHAPTER 4. ERGODICITY OF RECURSIVE CHAINS

1. General criteria of ergodicity for recursive chains

As it was noticed in Chapter 2, the ergodicity conditions for SRS and MC are close both formally and in essence. This relationship is exposed even more explicitly while comparing RC and MC.

Let $\{X(n)\}$, $X(0) = const$ be a RC with driver $\{\xi_n\}$. Let us fix a number $m \geq 0$ and denote by $\mathbf{F}_{n,m}$ the $\sigma-$algebra generated by the random variables $\{X(1), \dots, X(n); \{\xi_k; k \leq n+m\}\}$. As before, denote by $\mathbf{F}_n^{\xi}$ the $\sigma-$algebra generated by the variables $\{\xi_k; k \leq n\}$ so that $\mathbf{F}_{n,m} = \sigma(\mathbf{F}_n, \mathbf{F}_{n+m}^{\xi})$, $\mathbf{F}_{n,0} = \mathbf{F}_n$.

Theorem 1. *Assume that $\{X(n)\}$ is a RC with driver $\{\xi_n\}$ and that it satisfies* (3.5); *the sequence $\{\xi_n\}$ is stationary and metrically transitive. Let there exists for some $m \geq 0$ a stationary sequence of events $\{A_n\}$, $A_n \in \mathbf{F}_{n+m}^{\xi}$, such that*

(I R) $\mathbf{P}(A_0) > 0$;

(II R) *for $\omega \in A_n$ and $B \in \mathbf{B}_{\mathbf{X}}$ holds*

$$\mathbf{P}(X(n+m+1) \in B \mid \mathbf{F}_{n+m}) = \varphi(\xi_n, \dots, \xi_{n+m}; B) \quad a.s.,$$

where $\varphi : \mathbf{Y}^{m+1} \times \mathbf{B}_{\mathbf{X}} \to [0,1]$ is a measurable function; $\varphi(y_0, \dots, y_m; B)$ is a probability measure a.e. in $(y_0, \dots, y_m) \in \mathbf{Y}^{m+1}$ with respect to the distribution of $(\xi_0, \dots, \xi_m)$.

Then one can define on the same probability space with $\{X(n), \xi_n\}$ a stationary sequence $\{X^n\}$ such that $X(n) \xrightarrow{sc} X^n$. In addition, the sequence $\{X^n\}$ forms a RC, $\{X(n)\}$ and $\{X^n\}$ have the same driver $\{\xi_n\}$ and the same transition function $\mathbf{P}(x, y, B)$.

The assertion of Theorem 1 implies necessarily that the joint distribution of the pair (X^0, ξ_0)

$$\pi(A, B) = \mathbf{P}(X^0 \in A, \xi_0 \in B), \quad A \in \mathbf{B}_{\mathbf{X}}, \ B \in \mathbf{B}_{\mathbf{Y}}$$

satisfies relations

$$\pi(A, \mathbf{Y}) = \int_{\mathbf{X}} \int_{\mathbf{Y}} \pi(dx, dy) \cdot \mathbf{P}(x, y, A), \quad A \in \mathbf{B}_{\mathbf{X}}. \tag{1}$$

Analogous relations take place for the joint distributions of the vectors $(X^0, \xi_0, \xi_1, \dots, \xi_k)$ for any $k \geq 0$. For instance, for $k = 1$ the joint distribution of the variables (X^0, ξ_0, ξ_1)

$$\pi(A, B_0, B_1) = \mathbf{P}(X^0 \in A, \xi_0 \in B_0, \xi_1 \in B_1), \quad A \in \mathbf{B}_{\mathbf{X}}, B_0, B_1 \in \mathbf{B}_{\mathbf{Y}}$$

satisfies relations

$$\pi(A, \mathbf{Y}, \mathbf{Y}) = \int_{\mathbf{X}} \int_{\mathbf{Y}} \int_{\mathbf{Y}} \pi(dx, dy_0, dy_1) \, \mathbf{P}_2(x, y_0, y_1, A),$$

where $\mathbf{P}_2(x, y_0, y_1, A) = \int_{\mathbf{X}} \mathbf{P}(x, y_0, dz)\,\mathbf{P}(z, y_1, A)$.

It is evident that if the sequence $\{\xi_n\}$ consists of independent identically distributed random variables, then the RC $\{X(n)\}$ forms a MC and satisfies equalities

$$\pi(A, B) = \pi(A)\cdot \mathbf{P}(\xi_0 \in B), \text{ where } \pi(A) = \mathbf{P}(X^0 \in A),$$

and equation (1) can be rewritten in the form

$$\pi(A) = \int_{\mathbf{X}} \pi(dx)\cdot \int_{\mathbf{Y}} \mathbf{P}(\xi_0 \in dy)\cdot \mathbf{P}(x, y, A) \equiv \int_{\mathbf{X}} \pi(dx)\cdot \mathbf{P}(x, A),$$

where

$$\mathbf{P}(x, A) = \int_{\mathbf{Y}} \mathbf{P}(\xi_0 \in dy)\cdot \mathbf{P}(x, y, A).$$

R e m a r k 1. In addition to Theorem 1 one can formulate the following assertions:

If Conditions (I R), (II R) *are satisfied with the same events* $\{A_n\}$ *and the function* φ *for any initial value* $X(0) = x \in V_0$ *from a certain set,* $V_0 \in \mathbf{B}_{\mathbf{X}}$, *then the distribution of the limiting sequence* $\{X^n\}$ *does not depend on* $X(0) = x \in V_0$ *also.*

If there exists an increasing sequence of sets $\{V_k\}$, $\cup V_k = \mathbf{X}$, *such that, for any* $k = 1, 2, \ldots$, *there exist a stationary positive sequence* $\{A_{n,k}\}$ *and a function* φ_k *satisfying Conditions* (I R), (II R) *for the RC* $\{X(n)\}$ *with an arbitrary initial value* $X(0) = x \in V_k$, *then the distribution of the limiting sequence* $\{X^n\}$ *does not depend on* $X(0) = x \in \mathbf{X}$.

Theorem 1 admits the following converse statement.

Theorem 2. *Let the stationary sequence* $\{X^n\}$ *be defined on the same probability space with the SRS* $\{X(n)\}$, *and let* $X(n) \xrightarrow{sc} X^n$. *Then there exist an integer* $m \geq 0$, *a measurable function* $g: \mathbf{Y}^{m+1} \to \mathbf{X}$, *and a stationary sequence of events* $\{A_n\}$, $A_n \in \mathbf{F}_{n+m}$, *where* A_n *are renovating events for* $\{X(n)\}$, *such that Conditions* (I R), (II R) *are satisfied for* $\{X(n)\}$ *with*

$$\varphi(\xi_n, \ldots, \xi_{n+m}; B) \equiv I(g(\xi_n, \ldots, \xi_{n+m}) \in B).$$

Theorem 2 is a reformulation of that part of Theorem 2.4, which deals with the necessity of existence of renovating events. In order to prove Theorem 1, it suffices to represent $\{X(n)\}$ in the form of a SRS with driver $\tilde{\xi}_n = (\xi_n, \alpha_n)$, where α_n are introduced in Theorem 3.3, and to notice that on the set A_n the random variable $X(n+m+1)$ is defined uniquely by the values of $\tilde{\xi}_n, \ldots, \tilde{\xi}_{n+m}$. The sequence $\{\tilde{\xi}_n\}$ is stationary and metrically transitive. Thus A_n form a stationary sequence of renovating events for the SRS $\{X(n)\}$ with the driver $\{\tilde{\xi}_n\}$. Hence the first part of Theorem 2.4 implies Theorem 1 and the assertions contained in Remark 1 follow from the reasoning introduced just below the statement of Theorem 2.3.

Theorem 1 can be extended in the following direction.

Theorem 3. *Let* $\{X(n)\}$ *be a RC with driver* $\{\xi_n\}$ *satisfying* (3.5); *the sequence* $\{\xi_n\}$ *being stationary and metrically transitive. Assume that there exist an integer* $m \geq 0$, *a* *stationary sequence of events* $\{A_n\}$, $A_n \in \mathbf{F}^{\xi}_{n+m}$ *, and measurable functions* $p : \mathbf{Y}^{m+1} \to$ $\to [0,1]$, $\varphi : \mathbf{Y}^{m+1} \times \mathbf{B}_{\mathbf{X}} \to [0,1]$ *(where* $\varphi(y_0, \dots, y_m; \cdot)$ *is a probability measure on* $\mathbf{X}$ *for a.e.* $(y_0, \dots, y_m) \in \mathbf{Y}^{m+1}$ *with respect to the distribution of* $(\xi_0, \dots, \xi_m)$ *) such that*

(I RC) $\mathbf{E}\{ I(A_0) \cdot p(\xi_0, \dots, \xi_m) \} > 0$; (2)

(II RC) *the inequality*

$$\mathbf{P}(X(n+m+1) \in B \mid \mathbf{F}_{n,m}) \geq p(\xi_n, \dots, \xi_{n+m}) \cdot \varphi(\xi_n, \dots, \xi_{n+m}; B) \qquad (3)$$

holds for $\omega \in A_n$, $B \in \mathbf{B}_{\mathbf{X}}$.

Then one can define on the same probability space with $\{X(n), \xi_n\}$ *a stationary sequence* $\{X^n\}$ *such that* $X(n) \xrightarrow{sc} X^n$ *. The sequence* $\{X^n\}$ *is a RC with the same driver and the same transition function as* $\{X(n)\}$.

It is clear that Theorem 1 is a particular case of Theorem 3 for $p(y_0, \dots, y_m) \equiv 1$ and satisfied (3) as equality. Thus Theorem 2 may be considered to be, in a certain sense, an extension of Theorem 3.

Let us also note that Condition (II) for MC is a particular case of Condition (II RC) for $\xi_n \equiv$ *const* .

R e m a r k 2. In the particular case when $m = 0$, $p \equiv$ *c o n s t*, and the measure φ does not depend on the variables $(y_0, \dots, y_m)$, Conditions (I RC) and (II RC) have the form

$$\mathbf{P}(A_0) > 0 \ ; \ \mathbf{P}(X(n+1) \in B \mid \mathbf{F}_n) \geq p \cdot \varphi(B) \ , \qquad (4)$$

for $\omega \in A_n \in \mathbf{F}^{\xi}_n$. By virtue of Theorem 3 these conditions provide for the ergodicity of the RC $\{X(n)\}$. Will this assertion remain valid if we assume that $A_n \in \mathbf{F}^{\xi}_{n-1}$ and replace the $\sigma-$algebra $\mathbf{F}_n$ in Condition (4) by a "poorer" $\sigma-$algebra $\sigma(X(n), \xi_{n-1}, \dots, \xi_0)$? The answer to this question is negative in the general case , which is illustrated by Example 2.5. Example 2.5 gives the negative answer also to another closely related question. From the view point of many applications, it would seem highly desirable (and rather natural, by analogy with MC) to try to obtain the ergodicity conditions of types (I RC) and (II RC), but for SRS and for a "poorer" $\sigma-$algebra in (3) (otherwise the left hand side in (3) is the indicator function). In the simplest case, $m = 0$, these conditions could have the form

$$\mathbf{P}(A_0 > 0) \ ; \ \mathbf{P}(X(n+1) \in B \mid X(n), \xi_{n-1}, \xi_{n-2}, \dots, \xi_0) \geq p \cdot \varphi(B)$$

for $\omega \in A_n \in \mathbf{F}^{\xi}_{n-1}$. However, as stated above, these conditions do not ensure ergodicity (see Example 2.5).

R e m a r k 3. Theorem 3 lacks an analog of Condition (III) of Chapter 2 (non-periodicity). This fact is evidently connected with the stationarity of the sequence $A_n, n \geq 0$.

Proof of Theorem 3. Let us note first of all that in the conditions of Theorem 3 one can assume that $p(y_0, \dots, y_m) \equiv$ *const* > 0 without loss of generality. Indeed, it follows from (2) that there exist a set $A^1_n \subseteq A_n$ and a number $p > 0$ such that $\mathbf{P}(A^1_n) > 0$ and

$p(\xi_n, \ldots, \xi_{n+m}) \geq p$ a.s. on A_n^1. Thus Condition (3) remains valid after A_n is replaced with A_n^1 and $p(\xi_n, \ldots, \xi_{n+m})$ with p.

So let us assume that $p = const > 0$. One may assume also that $p < 1$. Let us apply now a method introduced in [7], [8]; as in this publications, we reproduce our reasoning for $m = 0$ only for the sake of simplicity. Denote $\mathbf{P}(X(n), \xi_n, B) = \mathbf{P}(X(n+1) \in B \mid \mathbf{F}_{n,0})$.

We construct a sequence $\{\tilde{X}(n)\}$ on the same probability space with $\{\xi_n\}$ and a sequence $\{\delta_n\}$ of independent identically distributed variables, not depending on $\{\xi_n\}$ and having the distribution function $\mathbf{P}(\delta_n = 1) = 1 - \mathbf{P}(\delta_n = 0) = p$. The construction is as follows.

Assume that $\tilde{X}(0) = X(0) = const$. Let us specify some random variable δ_0, which is independent of $\{\xi_n\}$.

Let $\{\tilde{X}(k), \delta_k\}$ be already constructed for $k \leq n$. Denote $\xi_k^* = (\xi_k, \delta_k)$, $X^*(k) = (X(k), \delta_k)$. Define σ – algebras

$$\mathbf{F}_k^* = \sigma\{X^*(0), \ldots, X^*(k); \xi_0^*, \ldots, \xi_k^*\};$$

$$\mathbf{F}_k^{**} = \sigma\{X^*(0), \ldots, X^*(k); \xi_0^*, \ldots, \xi_k^*; \xi_{k+1}, \xi_{k+2}, \ldots\}.$$

We specify the pair $(\tilde{X}(n+1), \delta_{n+1})$ with the help of the conditional distribution

$$\mathbf{P}(\tilde{X}(n+1) \in B, \delta_{n+1} = 1 \mid \mathbf{F}_n^{**}) =$$

$$= \mathbf{P}(\tilde{X}(n+1) \in B \mid \mathbf{F}_n^{**}) \cdot \mathbf{P}(\delta_{n+1} = 1) = p \cdot \mathbf{P}(\tilde{X}(n+1) \in B \mid \mathbf{F}_n^{**}).$$

In other words, we consider $\tilde{X}(n+1)$ and δ_{n+1} to be conditionally independent with respect to $\mathbf{F}_n^{**}$; δ_{n+1} does not depend on $\mathbf{F}_n^{**}$.

Then with regard to (3.5) (or, similarly, (3.7)), we define $\mathbf{P}(\tilde{X}(n+1) \in B \mid \mathbf{F}_n^{**})$ from the equalities

$$\mathbf{P}(\tilde{X}(n+1) \in B \mid \mathbf{F}_n^{**}) = \mathbf{P}(\tilde{X}(n+1) \in B \mid \mathbf{F}_n^{*}) =$$

$$= \begin{cases} \varphi(\xi_n, B) & \text{for } \omega \in A_n, \delta_n(\omega) = 1; \\ \dfrac{1}{1-p}\,\mathbf{P}(\tilde{X}(n), \xi_n, B) - p \cdot \varphi(\xi_n, B) & \text{for } \omega \in A_n, \delta_n(\omega) = 0; \\ \mathbf{P}(\tilde{X}(n), \xi_n, B) & \text{for } \omega \notin A_n. \end{cases}$$

Note that (3.2) holds in this case and $\{\tilde{X}(n)\}$ forms a RC. Indeed, for $\omega \notin A_n$ this equality is valid by the definition, and for $\omega \in A_n$

$$\mathbf{P}(\tilde{X}(n+1) \in B \mid \tilde{X}(0), \ldots, \tilde{X}(n); \xi_0^*, \ldots, \xi_n^*) = p \cdot \varphi(\xi_n, B) +$$

$$+ (1-p)\frac{\mathbf{P}(\tilde{X}(n), \xi_n, B) - p \cdot \varphi(\xi_n, B)}{1-p} = \mathbf{P}(\tilde{X}(n), \xi_n, B).$$

Hence the distribution of the constructed sequence $\{\tilde{X}(n), \xi_n\}$ coincides with the initial distribution of $\{X(n), \xi_n\}$.

On the other hand, the RC $\{X^*(n)\}$ with driver $\{\xi_n^*\}$ and a stationary sequence of events $A_n^* = A_n \cap \{\delta_n = 1\}$ satisfy the conditions of Theorem 1. Indeed, $\mathbf{P}(A_n^*) = \mathbf{P}(A_n) \cdot p > 0$ and for

$$\mathbf{X}^* = \mathbf{X} \times \{0, 1\},\ \mathbf{B}^* = \mathbf{B}_{\mathbf{X}^*} \quad,\ B^* = (B_1, 1) \cup (B_2, 0)\ ,$$

while $\omega \in A_n^*$, holds the identity

$$\mathbf{P}(X^*(n+1) \in B^* \mid \mathbf{F}^*_{n,0}) = \varphi(\xi_n, B_1) \cdot p + \varphi(\xi_n, B_2)(1-p) \equiv$$

$$\equiv \varphi(\xi_n, B^*)\ .$$

The theorem is proved.

R e m a r k 4. We shall make use also of an assertion closely related to Theorem 3. Let $\{X(n)\}$ be a RC with driver $\{(\xi_n, \eta_n)\}$ taking values in arbitrary measurable spaces. Denote for a fixed $m \geq 0$ and arbitrary $n \geq 0$ by $\mathbf{F}'_{n,m}$ the $\sigma-$algebra generated by the random variables $\{X(1), \dots, X(n); \{\eta_k, k < n\}; \{\xi_k, k \leq n+m\}\}$, and by $\mathbf{F}''_{n,m}$ the $\sigma-$algebra generated by the variables $\{\{\eta_k, k < n\}; \{\xi_k, k \leq n+m\}\}$.

Corollary 1. *Let the stationary and metrically transitive sequence* $\{\xi_n\}$ *and the i.i.d. sequence* $\{\eta_n\}$ *be independent. Assume that there exists (for some integer* $m \geq 0$ *) a stationary sequence of events* $\{A_n\}$, $A_n \in \mathbf{F}''_{n,m}$ *such that Condition*s (I RC) *and* (II RC) *are satisfied with the* $\sigma-$*algebra* $\mathbf{F}_{n,m}$ *replaced by the* $\sigma-$*algebra* $\mathbf{F}'_{n,m}$. *Then the statement of Theorem* 3 *is also valid.*

Proof. Theorem 3.2 implies that $\{X(n)\}$ is also a RC with the driver $\{\xi_n\}$. Since the $\sigma-$algebras $\mathbf{F}^{\xi}_{n+m}$ and $\mathbf{F}^{\eta}_{n-1} \equiv \sigma(\eta_k, k < n)$ are independent and generate $\mathbf{F}''_{n,m} = \sigma(\mathbf{F}^{\xi}_{n+m}, \mathbf{F}^{\eta}_{n-1})$, the conditions of the corollary imply that there exist some sets $A'_n \in \mathbf{F}^{\xi}_{n+m}$, $A''_n \in \mathbf{F}^{\eta}_{n-1}$ such that $A'_n \cap A''_n \subseteq A_n$ and $\mathbf{P}(A'_n \cap A''_n) = \mathbf{P}(A'_n) \cdot \mathbf{P}(A''_n) > 0$. This means that the following relations,

$$\mathbf{P}(X(n+m+1) \in B \mid \mathbf{F}_{n,m}) =$$

$$= \mathbf{E}\{\mathbf{E}\{I(X(n+m+1) \in B \mid \mathbf{F}'_{n,m}\} \mid \mathbf{F}_{n,m}\} \geq$$

$$\geq \mathbf{E}\{\mathbf{E}\{I(X(n+m+1) \in B) \cdot I(A''_n) \mid \mathbf{F}'_{n,m}\} \mid \mathbf{F}_{n,m}\} \geq$$

$$\geq \mathbf{P}(A''_n)\, p(\xi_n, \dots, \xi_{n+m})\, \varphi(\xi_n, \dots, \xi_{n+m}; B)\ ,$$

hold a.s. on the set A'_n. Thus Theorem 3 can be applied. Corollary 1 is proved.

2. Ergodicity of RC with non-stationary drivers

One can formulate ergodicity theorems which are analogous to Theorems 2.5-2.7 for SRS also for the case of non-stationary drivers. For instance, Theorem 2.5 has the following analog.

Let $\{\xi_n\}$ and $\{\zeta_n\}$ be two drivers assuming values in the space $\mathbf{Y}$. Let us define two recursive chains: $\{X(n)\}$, $X(0) = const$, with the driver $\{\xi_n\}$ and the transition function $\mathbf{P}(x, y, B)$, and $\{Y(n)\}$, $Y(0) = const$, with the driver $\{\zeta_n\}$ and the same transition function $\mathbf{P}(x, y, B)$.

Theorem 4. *Let a stationary metrically transitive sequence $\{\xi_n\}$, a sequence $\{\zeta_n\}$, a transition function $\mathbf{P}(x, y, B)$, and a set $V_0 \in \mathbf{B_X}$ be such that:*

1) *Conditions* (I RC), (II RC) *are satisfied for* RC $\{X(n)\}$ *for a certain integer $m \geq 0$, a stationary "positive" renovating sequence $A_n \in \mathbf{F}^{\xi}_{n+m}$, functions p, φ, and for any initial condition $X(0) \in V_0$;*

2) *the sequence $\{\zeta_n\}$ c-converges to $\{\xi_n\}$;*

3) *the sequence $\{Y(n)\}$ satisfies the relations*

$$\mathbf{P}\Big(\bigcup_{k \geq n} \{Y(k) \in V_0\}\Big) = 1 \text{ for each } n .$$

Then one can define on the same probability space with $\{Y(n)\}$ a stationary sequence $\{X^n\}$ such that $Y(n) \xrightarrow{c} X^n$. Moreover, $\{X^n\}$ is a RC *with the driver $\{\xi_n\}$ and the transition function $\mathbf{P}(x, y, B)$.*

The proof of Theorem 4 practically duplicates the proof of Theorem 2.5 and we shall not present it here.

3. On certain conditions sufficient for ergodicity

Let $\{X(n)\}$ be, as above, a RC with driver $\{\xi_n\}$ satisfying (3.5). Let us consider the case when the driver itself satisfies a mixing condition of the form:

$$\mathbf{P}(\xi_{n+1} \in B \mid \xi_n, \xi_{n-1}, \dots) \geq q \cdot \Psi(B) \quad \text{a.s.,} \tag{5}$$

for some $q > 0$ and a probability measure Ψ on $(\mathbf{Y}, \mathbf{B}_{\mathbf{B_Y}})$.

Let us consider a sequence of independent identically distributed random variables $\{\hat{\xi}_n\}$ with the distribution Ψ, which may be defined, generally speaking, on another probability space. For any initial condition $\hat{X}(0) = x \in \mathbf{X}$ introduce a RC $\{\hat{X}(x, n)\}$ with driver $\{\hat{\xi}_n\}$ and the transition function $\mathbf{P}(x, y, B)$ of the initial RC. In accordance with the argument above, the RC $\hat{X}$ is a MC.

Theorem 5. *Let* (4) *be satisfied. Assume that there exist a set $V \subseteq \mathbf{X}$, a stationary sequence of events $A_n \in \mathbf{F}^{\xi}_{n-1}$, $\mathbf{P}(A_n) > 0$, a number $m \geq 0$, and measurable functions $p : \mathbf{Y}^{m+1} \to [0, 1]$, $\varphi : \mathbf{Y}^{m+1} \times \mathbf{B_X} \to [0, 1]$ such that $\mathbf{E}\, p(\hat{\xi}_0, \dots, \hat{\xi}_m) > 0$, $\varphi(y_0, \dots, y_m; \cdot)$ is a probability measure on $(\mathbf{X}, \mathbf{B_X})$ for any $(y_0, \dots, y_m) \in \mathbf{Y}^{m+1}$, and the following conditions are valid:*

1) $\mathbf{P}(X(n) \in V \mid \mathbf{F}^{\xi}_{n-1}) = 1$ *on the set A_n for all $n \geq 0$;*

2) $\mathbf{P}(\hat{X}(x, m+1) \in B \mid \hat{\xi}_0, \dots, \hat{\xi}_m) \geq p(\hat{\xi}_0, \dots, \hat{\xi}_m) \cdot \varphi(\hat{\xi}_0, \dots, \hat{\xi}_m; B)$

a.s. with respect to the distribution of $(\hat{\xi}_0, \dots, \hat{\xi}_m)$ for all $B \in \mathbf{B_X}$, $x \in V$.

Then the statement of Theorem 1 is valid for the RC $X(n)$.

Proof. Applying mathematical induction, we construct a supplementary sequence $\{ \tilde{\xi}_n \equiv (\xi_n^*, \hat{\xi}_n, \beta_n), n \geq 0 \}$ on some probability space in the following way. For $n = 0$ assume the random variables $\hat{\xi}_0$ and β_0 to be independent, $\hat{\xi}_0$ assuming values in the space $(\mathbf{Y}, \mathbf{B}_{\mathbf{B}_{\mathbf{Y}}})$ and having the distribution Ψ, while β_0 admits values 0 and 1, $\mathbf{P} (\beta_0 = 1) = = 1 - \mathbf{P} (\beta_0 = 0) = q$. The random variable ξ_0^* assumes values from the space $(\mathbf{Y}, \mathbf{B}_{\mathbf{Y}})$ and is defined by the relations: $\xi_0^* = \hat{\xi}_0$ on the set $\{ \beta_0 = 1 \}$and

$$\mathbf{P} (\xi_0^* \in B \mid \hat{\xi}_0, \beta_0) = \mathbf{P} (\xi_0^* \in B \mid \beta_0) = \frac{\mathbf{P} (\xi_0 \in B) - q \cdot \Psi (B)}{1 - q}$$

on the set $\{ \beta_0 = 0 \}$ for all $B \in \mathbf{B}_{\mathbf{Y}}$.

Let the random variables $\xi_k^*, \hat{\xi}_k, \beta_k$ be already defined for all $0 \leq k \leq n$. We proceed to define the variables $\xi_{n+1}^*, \hat{\xi}_{n+1}, \beta_n$. We shall assume $\hat{\xi}_{n+1}$ and β_{n+1} to be mutually independent and independent of $\{ \tilde{\xi}_k, k \leq n \}$. Besides, $\hat{\xi}_{n+1}$ has the distribution Ψ on $(\mathbf{Y}, \mathbf{B}_{\mathbf{Y}})$ and $\mathbf{P} (\beta_{n+1} = 1) = 1 - \mathbf{P} (\beta_{n+1} = 0) = q$. The random variable ξ_{n+1}^* is defined with the help of the random variables already defined by the relations: $\xi_{n+1}^* = \hat{\xi}_{n+1}$ on the set $\{ \beta_{n+1} = 1 \}$. On the set $\{ \beta_{n+1} = 0 \}$ we put

$$\mathbf{P} (\xi_{n+1}^* \in B \mid \tilde{\xi}_0 \in d\tilde{y}_0, \ldots, \tilde{\xi}_n \in d\tilde{y}_n ; \hat{\xi}_{n+1} \in d\hat{y}_{n+1} ; \beta_{n+1} = 0) =$$

$$= \mathbf{P} (\xi_{n+1}^* \in B \mid \xi_0^* \in dy_0, \ldots, \xi_n^* \in dy_n ; \beta_{n+1} = 0) =$$

$$= \frac{1}{1 - q} [\mathbf{P} (\xi_{n+1} \in B \mid \xi_0 \in dy_0, \ldots, \xi_n \in dy_n) - q \cdot \Psi (B)]$$

for $B \in \mathbf{B}_{\mathbf{Y}}$, $\tilde{y}_i = (y_i, \hat{y}_i, c_i) \in \mathbf{Y} \times \mathbf{Y} \times \{ 0, 1 \}$, $\hat{y}_{n+1} \in \mathbf{Y}$.

According to the theorem of measure extension, we can define a sequence $\{ \tilde{\xi}_n, n \geq 0 \}$ with the state space $(\tilde{\mathbf{Y}}^\infty, \mathbf{B}_{\tilde{\mathbf{Y}}^\infty})$, $\tilde{\mathbf{Y}} = \mathbf{Y} \times \mathbf{Y} \times \{ 0, 1 \}$, such that the joint distributions of its elements satisfy the above relations. It is easy to check that the sequence $\{ \tilde{\xi}_n, n \geq 0 \}$ is stationary and metrically transitive. Thus by virtue of the Kolmogorov theorem one can extend this sequence to a stationary one defined on the entire integer lattice of the real axis. In other words, one can assume that a stationary metrically transitive sequence $\{ \tilde{\xi}_n = (\xi_n^*, \hat{\xi}_n, \beta_n ; - \infty < n < \infty \}$ is defined, which satisfies the following conditions: for each n

a) the random variables $\hat{\xi}_{n+1}$ and β_{n+1} are mutually independent, the $\sigma-$algebras $\sigma(\hat{\xi}_{n+1}, \beta_{n+1})$ and $\tilde{\mathbf{F}}_n \equiv \sigma\{ \tilde{\xi}_k, k \leq n \}$ are also independent;

b) $\xi_{n+1}^* = \hat{\xi}_{n+1}$ a.s. on the set $\{ \beta_{n+1} = 1 \}$ and the equality

$$\mathbf{P} (\xi_{n+1}^* \in B \mid \tilde{\xi}_k \in d\tilde{y}_k, \ldots, \tilde{\xi}_n \in d\tilde{y}_n ; \hat{\xi}_{n+1} \in d\hat{y}_{n+1} ; \beta_{n+1} = 0) =$$

$$= \mathbf{P} (\xi_{n+1}^* \in B \mid \xi_k^* \in dy_k, \ldots, \xi_n^* \in dy_n ; \beta_{n+1} = 0) =$$

$$= \frac{1}{1 - q} [\mathbf{P} (\xi_{n+1} \in B \mid \xi_k \in dy_k, \ldots, \xi_n \in dy_n) - q \cdot \Psi (B)]$$

holds for $B \in \mathbf{B}_{\mathbf{Y}}$, $\{\xi_n\}$, $k \le l \le n$, $\hat{y}_{n+1} \in \mathbf{Y}$ and a.e. in $(\tilde{y}_k, \dots, \tilde{y}_n; \hat{y}_{n+1})$ with respect to the distribution of $(\tilde{\xi}_k, \dots, \tilde{\xi}_n; \hat{\xi}_{n+1})$ for any $k < n$.

It is easy to see that the sequences $\{\xi_n^*\}$ and $\{\xi_n\}$ have the same finite-dimensional distributions. Indeed, the equalities

$$\mathbf{P}(\xi_{n+1}^* \in B \mid \xi_0^* \in dy_0, \dots, \xi_n^* \in dy_n) =$$

$$= q \cdot \mathbf{P}(\xi_{n+1}^* \in B \mid \xi_0^* \in dy_0, \dots, \xi_n^* \in dy_n; \beta_{n+1} = 1) +$$

$$+ (1 - q) \cdot \mathbf{P}(\xi_{n+1}^* \in B \mid \xi_0^* \in dy_0, \dots, \xi_n^* \in dy_n; \beta_{n+1} = 0) =$$

$$= q \cdot \Psi(B) + (1 - q) \frac{\mathbf{P}(\xi_{n+1} \in B \mid \xi_0 \in dy_0, \dots, \xi_n \in dy_n) - q \cdot \Psi(B)}{1 - q} =$$

$$= \mathbf{P}(\xi_{n+1} \in B \mid \xi_0 \in dy_0, \dots, \xi_n \in dy_n)$$

hold for any $n \ge 0$ a.s. with respect to the distribution of $(\xi_0^*, \dots, \xi_n^*)$.

Let us now construct a RC $X^*(n)$ with the driver $\{\tilde{\xi}_n\}$ which satisfies (3.5) and the following relation:

$$\mathbf{P}(X^*(n+1) \in B \mid X^*(n), \tilde{\xi}_n) = \mathbf{P}(X^*(n+1) \in B \mid X^*(n), \xi_n^*) =$$

$$= \mathbf{P}(X^*(n), \xi_n^*, B) \text{ a.s.,}$$

where the transition function $\mathbf{P}(x, y, B)$ is that of the RC $X(n)$ with the driver $\{\xi_n\}$. The sequence $X^*(n)$ defined in this way is a RC with the driver $\{\tilde{\xi}_n\}$; simultaneously it is a RC with the driver $\{\xi_n^*\}$. Besides, the finite-dimensional distributions of the sequences $(X^*(n), \xi_n^*)$ and $(X(n), \xi_n)$ coincide. Thus we can consider $(X^*(n), \xi_n^*)$ as a realization of $(X(n), \xi_n)$ on some probability space and, hence, we can omit the superscript asterisk.

Consider a stationary sequence of events

$$\tilde{A}_n = A_n \cap \{\beta_{n+i} = 1; 0 \le i \le m\} \in \tilde{\mathbf{F}}_{n+m} ; \mathbf{P}(\tilde{A}_n) = \mathbf{P}(A_n) \cdot q^{m+1} > 0 .$$

For any n the sequence $X(n+k), 0 \le k \le m$ has the same conditional distribution on the event $\tilde{A}_n$ as the Markov chain $\hat{X}(n+k), 0 \le k \le m$, with the "initial" condition at time n : $\hat{X}(n) = X(n) \in V$. Condition 2) of the theorem yields for $B \in \mathbf{B}_{\mathbf{Y}}$ the relations

$$\mathbf{P}(X(n+m+1) \in B \mid X(n) \in dx; \hat{\xi}_{n+i} \in dy_i, \beta_{n+i} = 1; 0 \le i \le m) =$$

$$= \mathbf{P}(\hat{X}(x, m+1) \in B \mid \xi_i \in dy_i; 0 \le i \le m) \ge p(y_0, \dots, y_m) \cdot \varphi(y_0, \dots, y_m; B)$$

a.e. in $x \in V$ with respect to the distribution of $X(n)$ and a.e. in $(y_0, \dots, y_m)$ with respect to the distribution of $(\hat{\xi}_0, \dots, \hat{\xi}_m)$. Hence a.e. in $\omega \in \tilde{A}_n$ hold the relations,

$$\mathbf{P}(X(n+m+1) \in B \mid \tilde{\mathbf{F}}_{n,m}) = \mathbf{P}(X(n+m+1) \in B \mid X(n); \xi_n, \dots, \xi_{n+m}) =$$

$$= \mathbf{P}(X(n+m+1) \in B \mid X(n); \hat{\xi}_n, \dots, \hat{\xi}_{n+m}) \geq$$

$$\geq p(\hat{\xi}_n, \dots, \hat{\xi}_{n+m}) \cdot \varphi(\hat{\xi}_n, \dots, \hat{\xi}_{n+m}; B),$$

where $\tilde{\mathbf{F}}_{n,m}$ is the σ-algebra generated by the random variables $X(1), \dots, X(n)$ and $\{\tilde{\xi}_k, k \leq n\}$.

Let us introduce the functions $\tilde{p}: \tilde{\mathbf{Y}}^{m+1} \to [0,1]$ and $\tilde{\varphi}: \tilde{\mathbf{Y}}^{m+1} \times \mathbf{B}_{\mathbf{X}} \to [0,1]$ according to the rule: for $\tilde{y}_l = (y_l, \hat{y}_l, c_l) \in \tilde{\mathbf{Y}}$, $0 \leq l \leq m$, $B \in \mathbf{B}_{\mathbf{X}}$

$$\tilde{p}(\tilde{y}_0, \dots, \tilde{y}_m) = p(\hat{y}_0, \dots, \hat{y}_m)$$

and

$$\tilde{\varphi}(\tilde{y}_0, \dots, \tilde{y}_m; B) = \varphi(\hat{y}_0, \dots, \hat{y}_m; B).$$

It follows from the conditions of the theorem and the relations introduced above that for the RC $X(n)$ with the driver $\{\tilde{\xi}_n\}$

$$\mathbf{E}[I(\tilde{A}_n) \cdot \tilde{p}(\tilde{\xi}_n, \dots, \tilde{\xi}_{n+m})] = \mathbf{E}[I(\tilde{A}_n) \cdot p(\hat{\xi}_n, \dots, \hat{\xi}_{n+m})] = \mathbf{P}(\tilde{A}_n) \cdot \mathbf{E}\, p(\hat{\xi}_n, \dots, \hat{\xi}_{n+m}) > 0,$$

and for $B \in \mathbf{B}_{\mathbf{X}}$

$$\mathbf{P}(X(n+m+1) \in B \mid \tilde{\mathbf{F}}_{n,m}) \geq \tilde{p}(\tilde{\xi}_n, \dots, \tilde{\xi}_{n+m}) \cdot \varphi(\tilde{\xi}_n, \dots, \tilde{\xi}_{n+m}; B)$$

a.s. on the set $\tilde{A}_n$. Hence, the conditions of Theorem 3 are satisfied. Theorem 5 is proved.

E x a m p l e 1. Let $\{w(n)\}$, $w(0) = 0$ be the sequence of workload vectors in a multiserver queueing system $G/G/l$:

$$w(n+1) = R(w(n) + e_1 s_n - i\tau_n)^+, \tag{6}$$

where s_n are service times, τ_n are inter-arrival times, the sequence $\xi_n \equiv (s_n, \tau_n)$ is stationary and metrically transitive; $e_1 = (1, 0, \dots, 0)$, $i = (1, 1, \dots, 1)$, and R is the permutation of coordinates of l-dimensional vectors to non-decreasing order. As demonstrated in [2], if

$$l \cdot \mathbf{E}\,\tau_n > \mathbf{E}\, s_n, \tag{7}$$

one can construct a stationary sequence $\{Y^n\}$ of l-dimensional vectors such that $w(n) \leq Y^n$ a.s. for all n. Moreover, it is demonstrated in [2, p.360] that, if $\mathbf{P}(\tau_{n+1} > x \mid \mathbf{F}_n^{\xi}) > 0$ a.s. for all x, where $\mathbf{F}_n^{\xi} = \sigma\{s_k, \tau_k; k \leq n\}$, then the sequence $\{w(n)\}$ sc-converges to some non-singular stationary sequence $\{w^n\}$ under some additional assumptions.

SRS being a particular case of RC, Theorems 1-5 are applicable to SRS as well (though "singularly", since the event indicators should be placed both on the left-hand and the right-hand sides of inequality (II RC)). In particular, Theorem 5 provides another version of ergodicity conditions for the SRS $\{w(n)\}$.

Corollary 2. *Assume that* (7) *is satisfied for the sequence* $w(n)$ *of the form* (6). *If, moreover, there exist a number* $q > 0$ *and a probability measure* Ψ *on* $[0, \infty) \times [0, \infty)$ *such that*

1) $\mathbf{P}((\tau_{n+1}, s_{n+1}) \in B \mid \mathbf{F}_n^{\xi}) > q \cdot \Psi(B)$ *a.s. for all Borel* $B \subseteq [0, \infty) \times [0, \infty)$,

and

2) $\inf\{z : \Psi(C_z) > 0\} < 0$, *where* $C_z = \{(x, y) : y - lx \le z\}$,

then the statement of Theorem 1 *holds for the* RC $\{w(n)\}$.

Proof. Let us find such $z_0 < 0$ that $\Psi(C_{z_0}) > 0$, and such numbers $y_0 \ge 0$ and $x_0 > 0$ that $y_0 - lx_0 < z_0$ and $\Psi(D) > 0$, where $D = [0, y_0] \times [x_0, \infty)$. Define a number $\hat{q} = q \cdot \Psi(D) > 0$ and a measure $\hat{\Psi}(B) = \Psi(B \cap D) / \Psi(D)$. Note that Condition (3) remains valid after replacement of q and Ψ by $\hat{q}$ and $\hat{\Psi}$ respectively.

Let us choose a number M large enough for the event $A_n = \{Y_n \le (M, \dots, M)\}$ to have positive probability. Let $(\hat{\tau}_n, \hat{s}_n)$ be a sequence of i.i.d. two-dimensional random vectors with the distribution $\hat{\Psi}$ (here $l \cdot \hat{\tau}_n > \hat{s}_n$ a.s. and, in particular, $l \cdot \mathbf{E}\hat{\tau}_n > \mathbf{E}\hat{s}_n$). Consider two l – server queueing systems with the same driver $\hat{\xi}_n = (\hat{\tau}_n, \hat{s}_n)$ and different initial conditions, $\hat{w}_0(0) = (0, \dots, 0)$ and $\hat{w}_M(0) = (M, \dots, M)$. As demonstrated in [18], for any fixed $M < \infty$ the sequences $\{\hat{w}_0(n)\}$ and $\{\hat{w}_M(n)\}$ sc-converge to the same stationary sequence $\{\hat{w}^n\}$. Denote by $\gamma = \gamma(M)$ the "coupling" time of the sequences $\{\hat{w}_0(n)\}$ and $\{\hat{w}_M(n)\}$: $\gamma = \min\{n \ge 0 : \hat{w}_0(n) = \hat{w}_M(n)\}$, and find such m that $\mathbf{P}(\gamma \le m) > 0$.

Let $x = (x_1, \dots, x_l) \le (M, \dots, M)$ be any vector with non-negative coordinates and $\hat{w}_x(n), n \ge 0$, be the sequence of workload vectors in a l – server queueing system with the driver $\hat{\xi}_n = (\hat{\tau}_n, \hat{s}_n)$ and the initial condition $\hat{w}_x(0) = x$. Inequalities $\hat{w}_0(n) \le \le \hat{w}_x(n) \le \hat{w}_M(n)$ are known to be preserved a.s. for any $n \ge 0$ (see, e.g., [18]). Thus equalities $\hat{w}_x(m+1) = \hat{w}_0(m+1)$ hold for any $x \le (M, \dots, M)$ on the set $\{\gamma \le m\}$. Hence the conditions of Theorem 5 are satisfied for $p(\hat{\xi}_0, \dots, \hat{\xi}_m) = I(\gamma \le m)$ and $\varphi(\hat{\xi}_0, \dots, \hat{\xi}_m; B) = I(\hat{w}_0(m+1) \in B)$. Corollary 2 is proved.

Let us go back to an arbitrary RC $\{X(n)\}$ with driver $\{\xi_n\}$ and assume the driver $\{\xi_n\}$ to satisfy a mixing condition which is stronger than (4):

$$\mathbf{P}(\xi_{n+1} \in B;\ (\xi_{n+2}, \xi_{n+3}, \dots) \in C \mid \xi_n, \xi_{n-1}, \dots) \ge$$
$$\ge q \cdot \Psi(B) \cdot \mathbf{P}((\xi_{n+2}, \xi_{n+3}, \dots) \in C \mid \xi_n, \xi_{n-1}, \dots) \text{ a.s.} \tag{8}$$

for some number $q > 0$, probability measure Ψ on $(\mathbf{Y}, \mathbf{B}_{\mathbf{Y}})$, each $B \in \mathbf{B}_{\mathbf{X}}$ and each cylindrical set $C \in \mathbf{Y}^{\infty}$. Then the following theorem is valid.

Theorem 6. *Let* (8) *be satisfied. Assume that there exist a set* $V \subseteq \mathbf{X}$, *a stationary sequence of events* $A_n \in \mathbf{F}_{n-1}^{\xi}$, $\mathbf{P}(A_n) > 0$, *numbers* $m \ge 0$, $p > 0$ *and a probability measure* φ *on* $(\mathbf{X}, \mathbf{B}_{\mathbf{X}})$ *such that*

1) $\mathbf{P}(X(n) \in V \mid \mathbf{F}_{n-1}^{\xi}) = 1$ *on the set* A_n *for all* $n \ge 0$,

2) $\mathbf{P}(\hat{X}(x, m+1) \in B) \ge p \cdot \varphi(B)$ for all $B \in \mathbf{B}_{\mathbf{X}}$, $x \in V$.

Then the statement of Theorem 1 *holds for the* RC $X(n)$.

Note that Condition (7) is equivalent to the following one:

$$\mathbf{P}(\xi_n \in B \mid \{\xi_k; k \ne n\}) \ge q \cdot \Psi(B) \text{ a.s.} \tag{9}$$

for all $B \in \mathbf{B}_{\mathbf{X}}$. Moreover, it is sufficient for (8) or (9) that the mixing coefficient of the form

$$h \equiv \inf (\mathbf{P} (A \cap B) \mid \mathbf{P} (A) \cdot \mathbf{P} (B))$$

be positive (where the infimum is taken over all the sets $A \in \mathbf{F}^{\xi}_{-\infty,0}$ and $B \in \mathbf{F}^{\xi}_{1,\infty}$ of positive probability).

Proof of Theorem 6. Let $\{ \xi_n \}$ be a stationary metrically transitive sequence satisfying (8). By means of a mathematical induction argument we construct a supplementary sequence $\tilde{\xi}_n = (\xi_n^* , \xi_n^{(1)} , \xi_n^{(0)} , \beta_n)$ on the same probability space with $\{ \xi_n \}$ in the following way. For $n = 0$ we assume that the random variables $\xi_0^{(1)}$ and β_0 are mutually independent and do not depend on $\{ \xi_i \}$, the variable $\xi_0^{(1)}$ has the distribution Ψ on space $(\mathbf{Y}, \mathbf{B}_{\mathbf{Y}})$ and $\mathbf{P} (\beta_0 = 1) = 1 - \mathbf{P} (\beta_0 = 0) = q$. The random variable $\xi_0^{(0)}$ does not depend on $\xi_0^{(1)}$ and β_0 ; it is determined according to the conditional distribution

$$\mathbf{P} (\xi_0^{(0)} \in B \mid \{ \xi_i , i \neq 0 \}) = \frac{1}{1 - q} [\mathbf{P} (\xi_0 \in B \mid \{ \xi_i ; i \neq 0 \}) - q \cdot \Psi (B)].$$

The joint distribution of the variables ξ_0 and $\xi_0^{(0)}$ may be arbitrary. For the sake of definiteness one may consider the variables ξ_0 and $\xi_0^{(0)}$ to be conditionally independent with respect to $\{ \xi_i , i \neq 0 \}$. The random variable ξ_0^* is defined by the equality

$$\xi_0^* = \xi_0^{(1)} \cdot I (\beta_0 = 1) + \xi_0^{(0)} \cdot I (\beta_0 = 0) .$$

Let us introduce the sequence $\{ \xi_n (0) ; - \infty < n < \infty \}$ assuming $\xi_0 (0) = \xi_0^*$ and $\xi_n (0) = \xi_n$ for $n \neq 0$. It is easily seen that the finite-dimensional distributions of the sequences $\{ \xi_n (0) \}$ and $\{ \xi_n \}$ coincide.

Suppose the random variables $\tilde{\xi}_l = (\xi_l^* , \xi_l^{(1)} , \xi_l^{(0)} , \beta_0)$ and the sequences $\{ \xi_n (l) , - \infty < n < \infty \}$ to be constructed for all $0 \leq l \leq k$, where

$$\xi_n (l) = \begin{cases} \xi_n^* & \text{for } o \leq n \leq l , \\ \xi_n & \text{for } n > l \text{ or } n < 0 . \end{cases}$$

The sequences $\{ \xi_n (l) , - \infty < n < \infty \}$ have the same finite-dimensional distributions as the initial sequence $\{ \xi_n \}$. Select the random variables $\tilde{\xi}_{k+1} = (\xi_{k+1}^* , \xi_{k+1}^{(1)} , \xi_{k+1}^{(0)} , \beta_{k+1})$ according to the following conditions: the variables $\xi_{k+1}^{(1)}$ and β_{k+1} are mutually independent and do not depend of the random variables defined previously, moreover, $\xi_{k+1}^{(1)}$ has the distribution Ψ on $(\mathbf{Y}, \mathbf{B}_{\mathbf{Y}})$, and $\mathbf{P} (\beta_{k+1} = 1) = 1 - \mathbf{P} (\beta_{k+1} = 0) = q$; the random variable $\xi_{k+1}^{(0)}$ does not depend on $\xi_{k+1}^{(1)} , \beta_{k+1}$ and has the conditional distribution

$$\mathbf{P} (\xi_{k+1}^{(0)} \in B \mid \{ \xi_i^{(k)} , i = k + 1 \} , \{ \xi_j^{(0)} , \xi_j^{(1)} , \beta_j ; j \leq k \}) =$$

$$= \frac{1}{1 - q} [\mathbf{P} (\xi_{k+1} \in B \mid \{ \xi_i (k) ; i \neq k + 1 \}) - q \cdot \Psi (B)] \text{ a.s.}$$

The random variable ξ^*_{k+1} is defined by the equality:

$$\xi^*_{k+1} = \xi^{(1)}_{k+1} \cdot I(\beta_{k+1} = 1) + \xi^{(0)}_{k+1} \cdot I(\beta_{k+1} = 0) .$$

Define the sequence $\{\xi_n(k+1); -\infty < n < \infty\}$ as follows: $\xi_{k+1}(k+1) = \xi^*_{k+1}$ and $\xi_n(k+1) = \xi_n(k)$ for $n \neq k+1$.

Therefore, for any $0 \leq k < \infty$ we have defined a stationary metrically transitive sequence $\{\tilde{\xi}_l = (\xi^*_l, \xi^{(1)}_l, \xi^{(0)}_l, \beta_l); o \leq l \leq k\}$ possessing the following properties:

a) the finite-dimensional distributions of the sequence $\{\xi^*_l; 0 \leq l \leq k\}$ are those of the sequence $\{\xi_l; 0 \leq l \leq k\}$;

b) for any $0 \leq l \leq k$ the random variables $\xi^{(1)}_l$ and β_l are mutually independent and do not depend on the random variables $\{\tilde{\xi}_i; o \leq i \leq k, i \neq l\}$, while $\xi^{(1)}_l$ has the distribution Ψ on the space $(\mathbf{Y}, \mathbf{B_Y})$ and $\mathbf{P}(\beta_l = 1) = 1 - \mathbf{P}(\beta_l = 0) = q$;

c) $\xi^*_l = \xi^{(1)}_l \cdot I(\beta_l = 1) + \xi^{(0)}_l \cdot I(\beta_l = 0)$.

According to the theorem of measure extension we can define the sequence $\{\tilde{\xi}_n, n \geq 0\}$ assuming values in the space $(\tilde{\mathbf{Y}}^\infty, \mathbf{B}_{\tilde{\mathbf{Y}}^\infty})$, $\tilde{\mathbf{Y}} = \mathbf{Y} \times \mathbf{Y} \times \mathbf{Y} \times \{0, 1\}$, with the finite-dimensional distributions satisfying the relations introduced above. The sequence obtained being stationary, it can be extended to a stationary one on the entire axis according to the Kolmogorov theorem on compatible distributions.

Then, following the proof of Theorem 5, one can define a sequence $\{X^*(n)\}$ which is a RC with driver $\{\xi^*_n\}$ such that the finite-dimensional distributions of $\{X^*(n), \xi^*_n\}$ and $\{X(n), \xi_n\}$ coincide. Thus one may consider $\{X^*(n), \xi^*_n\}$ as a realization of $\{X(n), \xi_n\}$ on some probability space and omit the superscript asterisk further on.

Let us define the $\sigma-$ algebras

$$\mathbf{F}'_{n,m} = \sigma\{X(1), \dots, X(n), \{\xi^{(1)}_j, j < n\}, \quad \{(\xi^{(0)}_j, \beta_j), j \leq n+m\}\},$$

$$\mathbf{F}''_{n,m} = \sigma\{\{\xi^{(1)}_j, j < n\}, \{(\xi^{(0)}_j, \beta_j), j \leq n+m\}\},$$

and consider the sets $\tilde{A}_n = A_n \cap \{\beta_{n+i} = 1; 0 \leq i \leq m\} \in \mathbf{F}''_{n,m}$ with the positive probability $\mathbf{P}(\tilde{A}_n) = \mathbf{P}(A_n) \cdot q^{m+1} > 0$.

For any $n \geq 0$ and $B \in \mathbf{B_X}$ the equality

$$\mathbf{P}(X(n+m+1) \in B \mid \mathbf{F}'_{n,m}) = \mathbf{P}(X(n+m+1) \in B \mid X(n); \beta_{i+n}, 0 \leq i \leq m)$$

holds a.s. on the set $\tilde{A}_n$ and

$$\mathbf{P}(X(n+m+1) \in B \mid X(n) \in dx; \beta_{n+i} = 1; 0 \leq i \leq m) =$$

$$= \mathbf{P}(\hat{X}(x, m+1) \in B) \geq p \cdot \varphi(B)$$

a.s. in $x \in V$ with respect to the distribution of $X(n)$. Thus the conditions of Corollary 1 for $\xi_n = (\xi^{(0)}_n, \beta_n)$, $\eta_n = \xi^{(1)}_n$ are satisfied. Theorem 6 is proved.

Let us go back to Example 1. Theorem 6 yields

Corollary 3. *Assume that the sequence* $\{w(n)\}$ *is defined by relations* (6), *and* (7) *holds. Let, moreover, there exist a number* $q > 0$ *and a probability measure* Ψ *on* $[0,\infty) \times [0,\infty)$ *such that*

1) (8) *is valid for the sequence* $\xi_n = (\tau_n, s_n)$,

2) *the measure* Ψ *contains an absolutely continuous component with respect to Lebesgue measure* λ *on* $[0,\infty) \times [0,\infty)$,

$$\Psi(B) \geq \iint_B g(x,y) \cdot \lambda(dx \times dy),$$

and, besides, there exist numbers $0 \leq a_1 < a_2 < \infty$ *and* $0 \leq b_1 < b_2 < \infty$ *such that*

$$\operatorname*{vrai\,inf}_A g(x,y) \equiv C > 0 ,$$

where $A = [a_1, a_2] \times [b_1, b_2]$.

Then the statement of Theorem 1 *is valid for the sequence* $\{w(n)\}$.

Proof. Define the number $\hat{q} = q \cdot C$ and the measure $\hat{\Psi}(B) = \lambda(B \cap A)$. The conditions of the corollary imply that (8) remains valid after replacement of q and Ψ respectively by $\hat{q}$ and $\hat{\Psi}$. Let us define the number $M > 0$ just as in the proof of Corollary 2, and, for arbitrary $0 \leq x = (x_1, \dots, x_l) \leq (M, \dots, M)$, consider the MC $\hat{w}_x(n)$ with the initial condition $\hat{w}_x(0) = x$. With the help of simple although cumbersome calculations one can ascertain that there exist a number $m \geq 0$, a set $C = [a'_1, a'_2] \times [b'_1, b'_2]$, $\lambda(C) > 0$, and a number $d > 0$ such that $\mathbf{P}(\hat{w}_x(m+1) \in B) \geq d \cdot \lambda(B)$ for all $B \subseteq C$, $x \leq (M, \dots, M)$. Thus the conditions of Theorem 6 are satisfied. Corollary 3 is proved.

CHAPTER 5. THE CONDITIONS PROVIDING EXISTENCE OF STATIONARY RENOVATING EVENTS. STATIONARY MAJORANTS. BOUNDEDNESS IN PROBABILITY

1. On the structure of renovating events

We have seen in Chapter 4 that Condition (II RC) for the ergodicity of RC is "brought to the level" of Condition (II) for MC. It is expressed in terms of local characteristics of the process under consideration, and in this sense it is final. Moreover, Condition (II RC) and its modifications are rather concise, and the verification of them in applied problems does not lead to any complications.

We face a different situation with Condition (I RC). It requires additional study. Relatively simple conditions sufficient for (I RC) are constructed below. In this case we observe once again an essential similarity with Condition (I) for MC.

Let us note, first of all, that inequalities of the form (II RC) are often fulfilled for RC, as well as for SRS, on events C_n^V of the form

$$C_n^V = \{X(n) \in V ;\ (\xi_n, \dots, \xi_{n+m}) \in Z\},$$
$$\mathbf{P}((\xi_n, \dots, \xi_{n+m}) \in Z) > 0 , \tag{1}$$

where $V \in \mathbf{B}_{\mathbf{X}}$ is a subset of the phase space $\mathbf{X}$, specified with the help of some (test) function $L : \mathbf{X} \to \mathbf{R}$,

$$V = V_N = \{ x : L(x) \le N \}. \tag{2}$$

It is evident that events of this form are not stationary in general. Only the "fragment" $\{ (\xi_n, \ldots, \xi_{n+m}) \in Z \}$ is stationary. Thus the required renovating events will be found if we succeed in constructing a stationary sequence B_n such that

$$B_n \subseteq \{ X(n) \in V \}; \ \mathbf{P}(B_n \cap \{ (\xi_n, \ldots, \xi_{n+m}) \in Z \}) > 0. \tag{3}$$

E x a m p l e 1 . Consider a sequence $\{ w(n) \}$, $w(0) = 0$ of workload vectors in a multiserver queueing system $G/G/l$ (see Example 4.1): $w(n+1) = R(w(n) + e_1 s_n - i \tau_n)^+$. It forms a SRS with driver (τ_n, s_n). Denote $w(n) = (w_{n1}, \ldots, w_{nl})$, where $w_{n1} \le w_{n2} \le \ldots \le w_{nl}$. The events

$$C_n^N = \{ \max_{1 \le i \le l} w_{ni} \le N ; s_{n+j} - l \tau_{n+j} \le -\varepsilon ; 0 \le j \le m \}$$

are known to be renovating for $\{ w(n) \}$ (see [2]) for any $\varepsilon > 0$ and sufficiently large N and $m = m(N, \varepsilon)$. In other words, we take for the set in (2)

$$V = V_N = \{ x : L(x) \le N \},$$

where $L(x) = \max_{1 \le i \le l} x_i$, and choose the set $Z \subseteq \mathbf{R}^{2m+2}$ to equal

$$Z = \{ (y_1, \ldots, y_{2m+2}) : y_{2k-1} \le l y_{2k} - \varepsilon ; 1 \le k \le m+1 \}.$$

The stationary events A_n implying C_n and satisfying (3) are constructed in [2].

Let us go back to arbitrary RC $\{ X(n) \}$.

Definition 1. A sequence of events $B_n \in \mathbf{F}_n$ is said to be *V-inducing for RC* $\{ X(n) \}$ (where $V \in \mathbf{B}_\mathbf{X}$), if

1) $\{ B_n \}$ is stationary, $\mathbf{P}(B_0) > 0$;

2) $B_n \subseteq \{ X(n) \in V \}$ for any $n \ge n_0$ for some $n_0 < \infty$.

The existence of V-inducing sequences for a rather wide class of the sets $\{ V \}$ is necessary for sc-convergence, which is demonstrated by the following

Lemma 1. *If* $\{ X(n) \} \xrightarrow{sc} \{ X^n \}$ *, then for any set V such that* $\mathbf{P}(X^0 \in V) > 0$ *, there exists a V - inducing sequence* $\{ B_n \}$.

Proof. Define, as before,

$$\nu = \min \{ n \ge 1 : U^{-k} X(k) = X^0 \text{ for all } k \ge n \}.$$

For a set V satisfying the conditions of the lemma, put

$$B_0 = \{ \nu \le N \} \cap \{ X^0 \in V \} , \ B_n = T^n B_0 ,$$

where

$$N = \min \{ l \ge 1 : \mathbf{P}(\nu \ge l) \ge 1 - \mathbf{P}(X^0 \in V) / 2 \}.$$

Then $\mathbf{P}(B_0) > 0$ and the inclusion

$$\{ X(n) \in V \} \supseteq \{ X(n) \in V \} \cap B_n = B_n$$

holds for $n \geq N$, since $X(n) = X^n$ a.s. on $T^n\{\nu \leq N\}$.

In regard to the definition introduced above, Theorem 4.3 implies

Corollary 1. *Let Condition* (4.2) *be satisfied a.s. on events* C_n^V *of the form* (1)-(2) *. If, moreover, there exists a* V *-inducing sequence* $n \geq N$ *, such that*

$$\mathbf{P}(B_0 \cap \{(\xi_0, \ldots, \xi_m) \in Z\}) > 0, \tag{4}$$

then the sequence $\{X(n)\}$ *sc-converges to the certain stationary sequence* $\{X^n \equiv U^n X^0\}$.

Condition (4) is evidently satisfied if $\mathbf{P}(B_0) + \mathbf{P}((\xi_0, \ldots, \xi_m) \in Z) > 1$ or if $\mathbf{P}(B_0) > 0$ and $\mathbf{P}((\xi_0, \ldots, \xi_m) \in Z \mid \mathbf{F}_{-1}) > 0$ a.e.

The above argument shows that one of the main problems arising in the course of study of Condition (I RC) is to find the ways to construct "attainable" V -inducing sequences of sufficiently large probability. If V is given in form (2), the problem is reduced to the construction of non-trivial stationary majorants L_n for the sequence $L(X(n))$: $L_n \geq L(X(n))$ a.s. If such a majorant L_n is constructed, V -inducing sequence has the form

$$B_n = \{L_n \leq N\} \subseteq \{L(X(n)) \leq N\} = \{X(n) \in V_N\}.$$

2. Conditions of the existence of stationary *V*-inducing events for the phase spaces $\mathbf{X} = [0, \infty)$ and $\mathbf{X} = (-\infty, \infty)$

In this section, as well as in the following one, we restrict ourselves to the consideration of SRS. Note that the assertions introduced below for SRS can be reformulated for RC, though in a more cumbersome form.

Let us consider a SRS $\{X(n)\}$ with values in $R_+ = [0, \infty)$. In this case equation (1.1) can be rewritten in the form

$$X(n+1) = (X(n) + h(X(n), \xi_n))^+, \tag{5}$$

where $x^+ = \max(0, x)$ and $h : \mathbf{R}_+ \times \mathbf{Y} \to \mathbf{R}$ is an arbitrary measurable function such that $(x + h(x, y))^+ = f(x, y)$. Recording SRS in the form (5) is sometimes more convenient, since it is less restrictive with respect to the increments $h(X(n), \xi_n)$.

In this case the set V often takes the form of a compact set $V = \{x : x \leq N\}$. Thus the construction of a V-inducing set is reduced to that of a stationary sequence $\{L_n\}$, which is a majorant for $\{X(n)\}$ in a natural sense,

$$L_n \geq X(n) \text{ a.s. for all } n \geq 0, \tag{6}$$

and to indicating such a value of N that $\mathbf{P}(L_n \leq N) > 1 - \mathbf{P}((\xi_0, \ldots, \xi_m) \in Z)$.

The following sufficient condition for (6) was introduced in [22] (see also Section 5):

Theorem 1. *Assume that there exist a number* $N > 0$ *and a function* $g_1 : \mathbf{Y} \to \mathbf{R}$ *possessing the properties*

1) $\mathbf{E}\, g_1(\xi_1) < 0$, (7)

2) $h(x, y) \leq \begin{cases} g_1(y) & \text{for } x > N, \\ g_1(y) + N - x & \text{for } x \leq N. \end{cases}$ (8)

If $X(0) \leq M < \infty$ *a.s., then the stationary sequence*

$$L_n = \max(M, N) + \max\left(0, \sup_{k\geq 1} \sum_{j=n-k}^{n-1} g_1(\xi_j)\right)$$

is a majorant for $\{X(n)\}$ *(in the sense of* (6)*).*

The following simple lemma will be rather helpful.

Lemma 2. *Conditions* (7)-(8) *are jointly equivalent to the following ones: there exist measurable functions* $g_2 : \mathbf{Y} \to \mathbf{R}$, $g_3 : \mathbf{Y} \to \mathbf{R}_+$ *and a constant* $C \geq 0$, *such that*

1) $\mathbf{E}\, g_2(\xi_1) < 0$ *;* $\mathbf{E}\, g_3(\xi_1) < \infty$, (9)

2) $X(n+1) - X(n) \leq g_2(\xi_n) + g_3(\xi_n) \cdot I(X(n) \leq C)$ (10)

a.s. for any $n \geq 0$.

Proof. Let Conditions (7)-(8) be fulfilled. Define a number $z > 0$ so that $\mathbf{E} \max(g_1(\xi_1), -z) < 0$. Then for $n \geq 0$

$$X(n+1) - X(n) = \max(-X(n), h(X(n), \xi_n)) \leq$$

$$\leq \max(-X(n), g_1(\xi_n)) + (N - X(n))^+ \leq \max(-z, g_1(\xi_n)) + z \cdot I(X(n) \leq z) +$$

$$+ N \cdot I(X(n) \leq N) \leq \max(-z, g_1(\xi_n)) + (N+z) \cdot I(X(n) \leq N+z) .$$

Hence (9)-(10) hold for $g_2(y) = \max(-z, g_1(y))$, $g_3(y) \equiv C = N + z$.

Let then (9)-(10) be satisfied. Denote $\alpha = -\mathbf{E}\, g_2(\xi_1)$ and find a number $M > 0$, such that $\mathbf{E}\{g_3(\xi_1); g_3(\xi_1) > M\} \leq \alpha/2$. Define the function $g_1(y) = g_2(y) + g_3(y) \cdot I(g_3(y) > M)$. Then $\mathbf{E}\, g_1(\xi_1) \leq -\alpha/2 < 0$ and

$$h(X(n), \xi_n) \leq X(n+1) - X(n) \leq g_1(\xi_n) + M \cdot I(X(n) \leq C) \leq$$

$$\leq g_1(\xi_n) + (M + C - X(n)) \cdot I(X(n) \leq M + C) .$$

Thus (8) holds for $N = M + C$. The lemma is proved.

Let us denote, as before, by $\mathbf{F}_n^{\xi}$ the $\sigma-$algebra generated by the sequence $\{\xi_k; k \leq n\}$, $\mathbf{F}^{\xi} = \mathbf{F}_{\infty}^{\xi}$. Note that the course of the proof of Theorem 1 would not be altered if we considered arbitrary stationary metrically transitive sequences $\{\Psi_n\}$ and $\{\varphi_n\}$ of random variables measurable with respect to $\mathbf{F}^{\xi}$, and if we also assumed $X(n)$ to admit values on the entire real line. Thus the following proposition is true under this new set of assumptions.

Corollary 2. *Assume that* $\mathbf{X} = (-\infty, \infty)$ *and let* $X(n)$, $X(0) \leq M < \infty$ *be a SRS with driver* $\{\xi_n\}$. *If there exist a number C and stationary sequences* $\{\Psi_n\}$, $\{\varphi_n\}$ *measurable with respect to* $\mathbf{F}^{\xi}$, *such that*

1) $\mathbf{E}\,\Psi_n < 0$, $\varphi_n \geq 0$ *a.s.,* $\mathbf{E}\,\varphi_n < \infty$, (11)

2) $X(n+1) - X(n) \leq \Psi_n + \varphi_n \cdot I(X(n) \leq C)$ *a.s.* (12)

for all n, then there exists an a.e. finite stationary majorant $\{L_n\}$ *for the sequence* $\{X(n)\}$.

R e m a r k 1. The statement of Corollary 2 would still be valid if the constant C in (12) were replaced by a random variable η_n, where $\{\eta_n\}$ is an arbitrary stationary sequence of $\mathbf{F}^{\xi}$-measurable random variables, $\mathbf{E}\,|\,\eta_n\,| < \infty$. Indeed, it is sufficient to introduce the sequence $Y(n) = X(n) - \eta_n^+$, $\eta_n^+ = \max(0, \eta_n)$ and notice that $Y(n+1) - Y(n) \leq \leq \Psi'_n + \varphi_n \cdot I(Y(n) \leq 0)$, where $\Psi'_n = \Psi_n + \eta_n^+ - \eta_{n+1}^+$ and $\mathbf{E}\,\Psi'_n = \mathbf{E}\,\Psi_n < 0$. Thus Corollary 2 is applicable to the sequence $\{Y(n)\}$ and so $X(n) \leq L'_n + \eta_n^+$, where $\{L'_n\}$ is a stationary majorant for $Y(n)$.

R e m a r k 2. It is easy to see that the statement of Corollary 2 would be still valid if we did not assume $\{X(n)\}$ to be a SRS, but considered $\{X(n)\}$ to be an arbitrary sequence defined on the same probability space with $\{\xi_n\}$ and satisfying (11)-(12).

Let us introduce some sufficient conditions for the construction of a stationary majorant, which are wider than (11)-(12). Note, first of all, that if a real-valued SRS $X(n+1) = f(X(n), \xi_n)$ satisfies Conditions (7)-(8), then for $G(y) = \sup_{z > N} (f(z, y) - z)$ the inequality $G(y) \leq g_1(y)$ is true for any $y \in \mathbf{Y}$ and hence $\mathbf{E}\,G(y) \leq \mathbf{E}\,g_1(\xi_1) < 0$. Let us provide an example for which $\mathbf{E}\,G(\xi_1) \geq 0$, but the construction of a stationary majorant still appears to be possible; it shoes that Conditions (7)-(8) (or (11)-(12)) are too restrictive.

E x a m p l e 3. Let the sequence $X(n+1) = (X(n) + h(X(n), \xi_n))^+$ assume values on the set $\{0, 1, 2, \ldots\}$ and $\xi_n = (\psi_n, \eta_n)$ be a two-dimensional driver with integer coordinates,

$$h(X(n), \xi_n) = \Psi_n + \begin{cases} 0 & \text{for even } X(n), \\ \chi_n & \text{for odd } X(n). \end{cases}$$

Suppose the sequences $\{\Psi_n\}$ and $\{\chi_n\}$ to be mutually independent, consisting each of i.i.d. random variables, moreover $\mathbf{P}(\Psi_n = 1) = 1 - \mathbf{P}(\Psi_n = -1) = p$, $0 < p < ½$ and $\mathbf{P}(\chi_n = 2) = \mathbf{P}(\chi_n = -2) = ½$. Note that $G(\xi_1) = \max(\Psi_1, \Psi_1 + \chi_1) = \Psi_1 + \max(0, \chi_1)$ holds for any $N > 0$, and $\mathbf{E}\,G(\xi_1) = 2p > 0$. On the other hand, let $X(0) = 0$. Denote $\mu_n = \max\{k \leq n : X(k) = 0\}$. Then for any $0 < \varepsilon < ½ - p$ and for $\Psi_k' = \Psi_k + \varepsilon$, $\chi_k' = \chi_k - \varepsilon$ hold the relations

$$X(n) = \sum_{k=\mu_n}^{n-1} \Psi_k + \sum_{k=\mu_n}^{n-1} \chi_k \cdot I\{k \text{ is odd}\} \leq \max\left(0, \sup_{k \leq n-1} \sum_{i=k}^{n-1} \Psi_i'\right) +$$

$$+ \max\left(0, \sup_{k \geq 0} \sum_{i=0}^{k} \chi'_{n-2i-2}\right) + \max\left(0, \sup_{k \geq 0} \sum_{i=0}^{k} \chi'_{n-2i-1}\right),$$

i.e., we have constructed a stationary majorant for $\{X(n)\}$.

Note that Condition (12) can be rewritten in the form

$$X(n+1) - X(n) \leq \begin{cases} \Psi_n & \text{for } X(n) > C, \\ \chi_n & \text{for } X(n) \leq C, \end{cases}$$

where $\chi_n = \Psi_n + \varphi_n$, $\mathbf{E}\,\chi_n^+ < \infty$.

In order to understand what extensions of (12) are possible let us consider the following model. Suppose that the interval (C, ∞) can be split into two parts $H_1 \cup H_2$ so that the inequalities

$$X(n+1) - X(n) \le \begin{cases} \Psi_{n,0} \text{ for } X(n) \le C; \\ \Psi_{n,1} \text{ for } X(n) \in H_1; \\ \Psi_{n,2} \text{ for } X(n) \in H_2 \end{cases} \tag{14}$$

are valid for some stationary metrically transitive sequences $\{\Psi_{n,0}\}$, $\{\Psi_{n,1}\}$, $\{\Psi_{n,2}\}$ measurable with respect to $\mathbf{F}^{\xi}$ and such that $\mathbf{E}\Psi_{n,0}{}^{+} < \infty$, $\mathbf{E}\Psi_{n,1} < 0$, $\mathbf{E}\Psi_{n,2} < 0$. Do these conditions imply the existence of a stationary majorant for $\{X(n)\}$? As illustrated by the following example, the answer to this question is in the negative. Moreover, in all these conditions $X(n)$ may go to infinity as $n \to \infty$.

Example 4. Let the sequence $X(n+1) = (X(n) + h(X(n), \xi_n))^{+}$ be defined by the driver $\xi_n = (\Psi_n, \chi_n)$ just as in the previous example, and let the sequence $\{\Psi_n\}$ consist of i.i.d. random variables, $\mathbf{P}(\Psi_n = 1) = 1 - \mathbf{P}(\Psi_n = -1) = p$. The sequence $\{\chi_n\}$ will be defined differently: $\{\chi_n\}$ does not depend on $\{\Psi_n\}$ and $\chi_{n+1} \equiv -\chi_n$, $\mathbf{P}(\chi_0 = 2) = \mathbf{P}(\chi_0 = -2) = 1/2$. Condition (14) is satisfied for $\Psi_{n,1} = \Psi_n$; $\Psi_{n,2} = \Psi_n + \chi_n$. However it is easy to ascertain that $X(n) \to \infty$ a.s. for any initial condition. For the sake of simplicity, let us check this fact for $X(0) = 0$ only and on the set $\{\chi_0 = 2\}$. It is easily seen that, if $\Psi_0 = 1$, then $X(1) = 1$ and $X(2) = (1 + \Psi_2 - 2)^{+} = 0$. Moreover, $X(2) = 0$ also in the case when $\Psi_0 = -1$, $\Psi_1 = -1$. Let $\nu = \min\{k \ge 0 : \Psi_{2k} = -1, \Psi_{2k+1} = 1\}$. It is clear that $\mathbf{P}(\nu < \infty) = 1$ and $X(2\nu) = 0$. Besides, $X(2\nu + 1) = 0$, $X(2\nu + 2) = 1$ and for $i \ge 1$ holds the equality

$$X(2\nu + 2i + 1) = 1 + 2i + \sum_{j=3}^{2i+1} \Psi_{2\nu + j - 1};$$

the right-hand side goes to infinity a.s. for $i \to \infty$.

Note that the "negative" effect employed in the example consists in the fact that $\mathbf{E}(\chi_{n+1} \mid \mathbf{F}_n^{\xi})$ may be positive with a positive probability. If we exclude this possibility, then under certain additional assumptions one can succeed in constructing a stationary majorant, as demonstrated below.

Let us proceed with the statement of results. We shall need one definition.

Let $V \subseteq \mathbf{X}$ be a measurable set. We shall say that *Condition* (N_V) holds if the random variables $X(n)$ assume only a finite number of values on the set V, i.e.,

(N_V) *There exists a finite collection of points* $x_1, \dots, x_M \in V$, *such that for all* n

$$\mathbf{P}(X(n) \in V) = \sum_{i=1}^{M} \mathbf{P}(X(n) = x_i).$$

Condition (N_V) is certainly satisfied if the set V is finite. If $\{X(n)\}$ assume values in the integer lattice $\{0, 1, 2, \dots\}$, then Condition (N_V) is satisfied for any bounded set V.

Theorem 2. *Let* $\mathbf{X} = \mathbf{R}_+$ *and* $\{ X(n) \}$, $X(0) = const$ *be a* SRS *with driver* $\{ \xi_n \}$. *Assume that for some* $C \geq 0$ *there exist functions* $F_1 : \mathbf{Y}^\infty \to \mathbf{R}$, $F_2 : \mathbf{R}_+ \times \mathbf{Y}^\infty \to \mathbf{R}$, ,$F_3 : \mathbf{Y}^\infty \to \mathbf{Y}$ *such that the random variables* $\Psi_n = F_1(\xi_n, \xi_{n-1}, \dots)$, $\zeta_n(x) = F_2(x, \xi_n, \xi_{n-1}, \dots)$, $\varphi_n = F_3(\xi_n, \xi_{n-1}, \dots)$ *satisfy the relations:*

1) $X(n+1) - X(n) \leq \Psi_n + \zeta_n(X(n)) + \varphi_n \cdot I(X(n) \leq C)$; (15)

2) $\mathbf{E}\,\varphi_n < \infty$, $\delta > 0$;

3) *for some* $\delta > 0$ $\sup_x \mathbf{E}\{ |\zeta_0(x)|^{2+\delta} \} < \infty$ *;*

4) *for all* $n \geq 0$ *and* x $\mathbf{E}\{ \zeta_n(x) \mid \mathbf{F}^{\xi}_{n-1} \} \leq 0$ *a.s.*

If in addition the set $V = [0, C]$ *satisfies Condition* (N_V), *then an a.e. finite stationary majorant can be constructed for the sequence* $\{ X(n) \}$.

Denote by $X(y, n)$ a SRS with initial condition $X(0) = y$, and denote by $\gamma_0(y)$ the first hitting time

$$\gamma_0(y) = \min\{ n \geq 1 : X(y, n) \in V \},$$

and by $S_n(y)$ the random variable

$$S_n(y) = \sum_{j=0}^{n} \zeta_j(X(y, j)).$$

Consider, along with (N_V), the weaker condition:

(N_V^1) *There exist a random variable* Φ_0, $\mathbf{E}\,\Phi_0 < \infty$, *and a finite collection of points* $x_1, \dots, x_M \in \mathbf{X}$, *such that for any* $y \in V, n \geq 1$ *the inequality*

$$S_n(y) \leq \Phi_0 + \max_{1 \leq i \leq M} S_n(x_i)$$

holds a.s. on the set $\{ \gamma_0(y) > n \}$.

Theorem 3. *If Condition* (N_V) *in Theorem* 2 *is replaced by Condition* (N_V^1), *then one can construct a stationary majorant for the sequence* $\{ X(n) = X(X(0), n) \}$ *with an arbitrary initial value* $X(0) \in \mathbf{R}_+$, $X(0) = const$.

According to the reasoning presented in the proof of Lemma 2, Theorems 2 and 3, one may assume without loss of generality that $\varphi_n \equiv C_1 = const \geq 0$. Let us note also that Corollary 2 follows from Theorem 3 for $\zeta_n \equiv 0$, $\Phi_0 \equiv 0$, $M = 1$, $x_1 = C$.

Proof of Theorem 2. Assume that U is, as before, the measure-preserving shift transformation generated by $\{ \xi_n \}$ and denote by $\mu_n = \max\{ k \leq n : X(k) \in V \}$ the last hitting time of the set $V = [0, C]$ on the interval $[0, n]$; $\mu_n = 0$ if $X(k) > C$ for all $k \leq n$.

The following relations hold for the sequence of the theorem

$$X(n+1) \leq C_2 + \sum_{j=\mu_{n+1}}^{n} \Psi_j + \sum_{j=\mu_{n+1}}^{n} \zeta_j(X(j)) \equiv$$

$$\equiv C_2 + \sum_{j=\mu_{n+1}}^{n} (\Psi_j + \alpha/2) + \sum_{j=\mu_{n+1}}^{n} (\zeta_j(X(j)) - \alpha/2) \equiv C_2 + \sum\nolimits_1 + \sum\nolimits_2 ,$$

where $\alpha = -\mathbf{E}\Psi_1 > 0$ and $C_2 = \max(x_0, C) + C_1$. Denote $\Psi_j{}' = \Psi_j + \alpha/2$;

$$\zeta_j{}'(y) = \zeta_j(y) - \alpha/2 \ ; \ S_n{}'(y) = \sum_{j=0}^{n} \zeta_j{}'(X(y,j)) \equiv S_n(y) - (n+1)\alpha/2 .$$

Note that $\sum_1 \le \max(0, \sup_{k\le n} \sum_{i=k}^{n} \Psi_j') \equiv \Psi^{n+1} < \infty$ a.s. and

$$\sum\nolimits_2 = I(\mu_{n+1} = 0) S_n{}'(x_0) +$$

$$+ \sum_{k=1}^{n} I(\mu_{n+1} = k) \sum_{i=1}^{M} I(X(k) = x_i) \sum_{j=k}^{n} \zeta_j{}'(X(j)) \equiv \sum\nolimits_3 + \sum\nolimits_4 ,$$

where

$$\sum\nolimits_3 \le U^{n+1} \max(0, \sup_{k\ge 0} U^{-k-1} S_k{}'(x_0)) \equiv T^{n+1}(x_0) ;$$

$$\sum\nolimits_4 = \sum_{k=1}^{n} I(\mu_{n+1} = k) \sum_{i=1}^{M} I(X(k) = x_i) \sum_{j=k}^{n} U^k \xi'_{j-k} (X(x_i, j-k)) \le$$

$$\le \sum_{i=1}^{n} I(\mu_{n+1} = k) \cdot \max_{1\le i\le M} U^k S'_{n-k}(x_i) \le$$

$$\le \sup_{k\le n} \max_{1\le i\le M} U^k S'_{n-k}(x_i) \equiv U^{n+1} (\max_{1\le i\le M} \sup_{k\ge 0} U^{-k-1} S_k{}'(x_i))$$

$$\equiv \max_{1\le i\le M} T^{n+1}(x_i) .$$

For each $i = 0, 1, \ldots, M$ the sequence $\{T^n(x_i)\}$ is stationary. Let us show that $T^n(x_i) < \infty$ a.s. Indeed,

$$\mathbf{P}(T^0(x_i) > t) \le \sum_{k=0}^{\infty} \mathbf{P}(U^{-k} S_k{}'(x_i) > t) =$$

$$= \sum_{k=0}^{\infty} \mathbf{P}(S_k{}'(x_i) > t) = \sum_{k=0}^{\infty} \mathbf{P}(S_k(x_i) > t + (k+1)\alpha/2).$$

If $\{S_n(x_i)\}$ forms a submartingale, Condition 3) and some well-known inequalities for submartingales [22] imply

$$\sum_{k=0}^{\infty} \mathbf{P}(S_k(x_i) > t + (k+1)\alpha/2) \le \sum_{k=1}^{\infty} k \cdot c(t + k\alpha/2)^{-2-\delta} ,$$

where the right-hand side goes to zero as $t \to \infty$. The theorem is proved.

Proof of Theorem 3 is obtained from the proof of Theorem 2, if we notice that in the conditions of Theorem 3 the upper estimate for $\sum_4$ coincides with

$$\tilde{T}^{n+1} \equiv U^n \left(\max_{1 \le i \le M} \sup_{k \ge 0} U^{-k} (\Phi_0 + S_k{}' (x_i)) \right)$$

and

$$\sum_{k=0}^{\infty} \mathbf{P} (S_k (x_i) + \Phi_0 > t + (k + 1) \alpha / 2) \le$$

$$\le \sum_{k=0}^{\infty} \mathbf{P} (S_k (x_i) > t / 2 + (k + 1) \alpha / 4) + \sum_{k=0}^{\infty} \mathbf{P} (\Phi_0 > t / 2 + (k + 1) \alpha / 4) .$$

3. Conditions for the existence of V-inducing events for an arbitrary phase space

The results stated in the preceding section for the phase spaces $\mathbf{X} = [0 , \infty)$ and $\mathbf{X} = (- \infty , \infty)$ are naturally transferred to the case of an arbitrary phase space $\mathbf{X}$ if we use the so-called test functions.

As in the preceding section, we restrict ourselves to the case of SRS.

Let $\{ X (n) \}$, $X (0) = const$, be a SRS with driver $\{ \xi_n \}$, assuming values in space $(\mathbf{X} , \mathbf{B_X})$, and $L : \mathbf{X} \to \mathbf{R}_+$ a measurable function. Denote $X^L (n) = L (X (n))$.

Corollary 3. *Assume that Conditions* (11)-(12) *are satisfied for the sequence* $\{ X^L (n) \}$. *Then one can construct for* $\{ X^L (n) \}$ *an a.e. finite stationary majorant* $\{ L_n \}$. *Hence if a number* $c' \ge c$ *is such that* $\mathbf{P} (L_n \le c') > 0$, *then the sequence of events* $B_n = \{ L_n \le c' \}$ *is* V' *-inducing for* $\{ X (n) \}$, *where* $V' = L^{-1} ([0 , c'])$.

Corollary 3 follows from Corollary 3, Remark 2 and the definition of V-inducing events.

The following theorem is also valid.

Theorem 4. *Let Conditions* 1)-4) *of Theorem* 2 *be satisfied for some* $c \ge 0$. *If, moreover, one of Conditions* (N_V) *or* (N_V^1) *is valid for the SRS* $\{ X (n) \}$ *and for the set* $V = L^{-1}([0 , C])$, *then one can construct an a.e. finite stationary majorant* $\{ L_n \}$ *for the sequence* $\{ X^L (n)\}$. *Thus the sequence of events* $B_n = \{ L_n \le c' \}$, *where* $c' \ge c$, $\mathbf{P} (B_n) > 0$, *is* V'*-inducing for* $\{ X (n) \}$ *if* $V' = L^{-1} ([0 , c'])$.

The assertion of Theorem 4 does not follow formally from Theorems 2 and 3, since the sequence $\{ X^L (n) \}$ does not, generally speaking, of necessity constitute a SRS. However, it is easy to make sure that the reasoning applied in the course of the proof of Theorems 2 and 3 can be applied also in this case without serious alterations.

Let us provide some examples showing that the conditions of Theorems 2-4 are similar to those in well-known criteria of positive recurrence for MC.

E x a m p l e 5. Let $\{ \xi_n \}$ be a stationary metrically transitive sequence with values on the line, $\mathbf{E} \xi_n < 0$, $\mathbf{E} (\xi_n{}^+)^2 < \infty$ and generate a new sequence using the function $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}_+$ and the initial condition $X (0) = 0 : X (n + 1) = f (X (n) , \xi_n)$.

Suppose that there exists a constant $c > 0$, such that the inequality $f (x , y) \le$ $\le (x + y / \max (x, c))^+$ is satisfied. Let us construct a stationary majorant for $\{ X (n) \}$.

Firstly, note that one may assume without loss of generality that $\mathbf{P}(\xi_n \geq -N) = 1$ for some very large N, $N > c^2$. In this case

$$f(x,y) \leq \begin{cases} N + y^+ / c & \text{for } 0 \leq x \leq N, \\ x + g_1(y)/x & \text{for } x > N, \end{cases}$$

where $g_1(y) = \max(y, -N)$. Secondly, introduce the test function $L(x) = x^2$. Then

$$L(f(x,y)) - L(x) \leq \begin{cases} (N + y^+ / c)^2, & \text{for } o \leq x \leq N, \\ 2g_1(y) + g_1^2(y)/x^2, & \text{else}. \end{cases}$$

Let us choose a number $k > N^2$ so that $\mathbf{E}(2g_1(\xi_1) + g_1^2(\xi_1)/k) < 0$ and define the function $g_2(y) = 2g_1(y) + g_1^2(y)/k$. Then

$$L(f(x,y)) - L(x) \leq \begin{cases} g_2(y) & \text{for } x^2 > k, \\ G(y) & \text{for } x^2 < k, \end{cases}$$

where $G(y) = (N + y^+/c)^2 + 2y^+ + \max(N^2, (y^+)^2)/k$, $\mathbf{E}G(\xi_1) < \infty$. Thus one can apply Corollary 3, i.e., there exists a stationary sequence L_n such that $L(X(n)) \leq L_n$ a.s. for all n. We obtain finally the estimate $X(n) \leq (L_n)^{1/2}$ for all n.

E x a m p l e 6. Let us consider the so-called oscillating random walk on the real line. Let $\{\xi_{n,1}\}, \{\xi_{n,2}\}, \{\xi_{n,0}\}$ be three mutually independent sequences consisting of independent random variables, identically distributed in each sequence, $\mathbf{E}|\xi_{n,0}| < \infty$, $\mathbf{E}\xi_{n,1} < 0$, $\mathbf{E}\xi_{n,2} < 0$. For $a, b > 0$ define the sequence $X(n)$ according to the rule

$$X(n+1) - X(n) = \begin{cases} \xi_{n,1} & \text{for } X(n) > b, \\ -\xi_{n,2} & \text{for } X(n) < -a, \\ \xi_{n,0} & \text{for } -a \leq X(n) \leq b. \end{cases}$$

For the sake of definiteness, assume $-\alpha \equiv \mathbf{E}\xi_{n,1} \geq \mathbf{E}\xi_{n,2}$. Let $\varphi_n = \xi_{n,1}$, $\eta_n = \xi_{n,2} - \xi_{n,1}$, $\mathbf{E}\eta_n \leq 0$. For a fixed number N define the random variables

$$\psi_n = \alpha/3 + \varphi_n + 2|\varphi_n| \cdot I\{|\varphi_n| > N\} + 2|\eta_n| \cdot I\{|\eta_n| > N\} + 2|\xi_{n,0}| \cdot I\{|\xi_{n,0}| > N\}$$

and choose N large enough in order to provide validity of relations

$$\mathbf{E}\psi_n \leq -\alpha/3, \quad \mathbf{E}\{\eta_n ; |\eta_n| \leq N\} \leq \alpha/3.$$

In this case

$$|X(n+1)| - |X(n)| \leq \psi_n + \zeta_n + \begin{cases} 3N, & \text{for } |X(n)| \leq N, \\ 0, & \text{else}, \end{cases}$$

where $\zeta_n = \zeta_n(X(n)) = \eta_n \cdot I\{|\eta_n| \leq N\} \cdot I\{X(n) < -a\} - \alpha/3$, i.e., Condition 1) of Theorem 2 is satisfied for $L(x) = |x|$. Conditions 2)- 4) of Theorem 2 are satisfied provided the values ζ_n are bounded. Last, Condition (N_V^1) is satisfied for $V = [-N, N]$, if we choose $\varepsilon << N$ so that ε divides N without a remainder, and set $M = 2N/\varepsilon + 1$, $x_i = -N + (i+1)\varepsilon$, $1 \le i \le M$. Hence Theorem 4 can be applied, i.e., there exists such a stationary sequence L_n for $\{X(n)\}$ that $|X(n)| \le L_n$ a.s. Note that the reasoning of this example can be applied in a somewhat more general case, when the sequence $\{(\xi_{n,0}; \xi_{n,1})\}$ is stationary and metrically transitive, and the sequence $\{\eta_n = \xi_{n,2} - - \xi_{n,1}\}$, being independent of $\{\xi_{n,0}; \xi_{n,1}\}$, consists of independent identically distributed random variables.

4. Conditions for boundedness in probability

Let us go back to the case of the phase space $\mathbf{X} = (-\infty, \infty)$ and study the following problem: what are the conditions for $\{X(n)\}$ to be bounded in probability, i.e., when $\sup_n \mathbf{P}(X(n) > t) \to 0$ for $t \to \infty$? Surely, the existence of a stationary majorant implies boundedness in probability. However the converse is not true in the general case, and boundedness in probability may be obtained in a much wider range of situations.

Note that although the problem of finding conditions for boundedness in probability lies aside from the main direction of the present paper, it is rather important in itself, since in applied problems it is frequently sufficient to indicate conditions providing for boundedness in probability of the process considered, while the study of stabilization conditions of the process is less important.

As we shall see below, conditions for boundedness in probability are much more general than those for the existence of V-inducing sets. Moreover, they are formulated for arbitrary sequences $\{X(n)\}$, which are not necessarily SRS or RC.

Theorem 5. *Let sequences $\{X(n)\}$, $\{\psi_n\}$, $\{\zeta_n\}$ of real-valued random variables and an increasing sequence of $\sigma-$ algebras $\mathbf{F}_n$ be defined on the same probability space so that*

1) $X(n+1) - X(n) \le \psi_n + \zeta_n + C_1 \cdot I(X(n) \le C_2)$ (16)

for some C_1, C_2 a.s. for all $n \ge 0$;

2) $\{\psi_n\}$ *is a stationary metrically transitive sequence;* $\mathbf{E}\,\psi_n < 0$;

3) $\mathbf{F}_n \supseteq \sigma\{\zeta_k; k \le n\}$; $\mathbf{E}(\zeta_{n+1} \mid \mathbf{F}_n) \le 0$ a.s.;

4) $\sup_n \mathbf{E}(|\zeta_n| \cdot g(|\zeta_n|)) \equiv C < \infty$ *for some function* $g : \mathbf{R}_+ \to \mathbf{R}_+$ *, which is continuous, concave, monotone increasing, and satisfies the conditions:*

$$g(0) = 0 \,, \quad \int_1^\infty [x \cdot g(x)]^{-1} dx < \infty \,.$$

Then the sequence $\{X(n)\}$ is bounded in probability.

It is evident that one may take $g(x) = x^\varepsilon$ with any $0 < \varepsilon \le 1$.

Corollary 4. *Assume that SRS $\{X(n)\}$ takes values in $\mathbf{R}_+ = [0, \infty)$, $X(0) = 0$, and the function f is monotone non-decreasing in its first variable. Then under conditions of Theorem* 5 *one can construct a stationary majorant for the sequence $\{X(n)\}$. Moreover, the distributions of $X(n)$ weakly converge to a non-singular limiting distribution.*

Proof of Corollary 4. The monotonicity properties show that the sequence $U^{-n}X(n)$ is non-decreasing. Hence there exists the a.s. limit $X^0 = \lim_{n\to\infty} U^{-n}X(n)$ and, by Theorem 5, $\mathbf{P}(X^0 > x) = \sup_n \mathbf{P}(U^{-n}X(n) > x) = \sup_n \mathbf{P}(X(n) > x) \to 0$ for $x \to \infty$.

Proof of Theorem 5. Let us begin with two auxiliary lemmas.

Lemma 3. *Let $g:\mathbf{R}_+ \to \mathbf{R}_+$ be a concave continuous non-decreasing function, $g(0)=0$. Then there exists a constant $K = K(g)$, $1 \le K \le 3$, such that for any numbers a, b*

$$|a+b| \cdot g(|a+b|) \le$$

$$\le |a| \cdot g(|a|) + K|b| \cdot g(|b|) + b(g(|a|) + |a| \cdot g'(|a|)) \operatorname{sign} a .$$

Here

$$g'(x) \equiv \lim_{\Delta \downarrow 0} \frac{g(x+\Delta) - g(x)}{\Delta}$$

is (for $x > 0$) the right derivative at the point x.

Proof. One may assume without loss of generality that $g(x) > 0$ for $x > 0$. Note then that $g'(x)$ is a non-increasing function, so the inequality $x \cdot g'(x) \le g(x)$ holds for any $x > 0$. Indeed,

$$g'(x) \cdot x = \int_0^x g'(x)\,dz = \lim_{\Delta\to 0} \int_\Delta^x g'(x)\,dz \le$$

$$\le \lim_{\Delta\to 0} \int_\Delta^x g'(z)\,dz = \lim_{\Delta\to 0} (g(x) - g(\Delta)) = g(x).$$

Thus one can introduce a constant $c = c(g) = \sup_{x>0} \frac{g'(x)\cdot x}{g(x)} \le 1$ and put $0 \cdot g'(0) = 0$.

It is sufficient to prove the assertion of the lemma for $a > 0$ only. Denote $d = |b|$ and consider three possible cases: 1) $b \ge 0$; 2) $b < 0$, $d \le a$; 3) $b < 0$, $d > a$.

Case 1). The equality

$$(a+b)\,g(a+b) = a \cdot g(a) + b \cdot g(b) + a[g(a+b) - g(a)] + b[g(a+b) - g(b)]$$

takes place, and, since

$$g(a+b) - g(a) = \int_a^{a+b} g'(z)\,dz \le g'(a) \cdot b \text{ and } g(a+b) - g(b) \le g(a),$$

one arrives, for any $K \ge 1$, at the estimate

$$(a+b) \cdot g(a+b) \le a \cdot g(a) + b \cdot g(b) + b[a \cdot g'(a) + g(a)] \le$$

$$\le a \cdot g(a) + Kb \cdot g(b) + b[a \cdot g'(a) + g(a)].$$

Case 2). The equality

$$(a+b)\,g(a+b) = (a-d)\,g(a-d) =$$

$$= a \cdot g(a) + d \cdot g(d) - a[g(a) - g(a-d)] - d[g(a-d) + g(d)]$$

takes place, and since

$$g(a-d) + g(d) \geq g(a) \text{ and } g(a) - g(a-d) = \int_{a-d}^{a} g'(z)\,dz \geq g'(a) \cdot d ,$$

the relations

$$(a-d)g(a-d) \leq a \cdot g(a) + d \cdot g(d) - a \cdot g'(a)d - d \cdot g(a) \leq$$
$$\leq a \cdot g(a) + K|b| \cdot g(|b|) + b(g(a) + a \cdot g'(a))$$

hold for any $K \geq 1$.

C a s e 3). The following relations are valid under the assumptions listed above:

$$|a+b| \cdot g(|a+b|) = (d-a) \cdot g(d-a) \leq d \cdot g(d) \leq$$
$$\leq a \cdot g(a) + (2+c) \cdot d \cdot g(d) - (1+c) \cdot d \cdot g(d) \leq$$
$$\leq a \cdot g(a) + (c+2) \cdot d \cdot g(d) - d \cdot (g(a) + a \cdot g'(a)) =$$
$$= a \cdot g(a) + (c+2)|b| \cdot g(|b|) + b(g(a) + a \cdot g'(a)) .$$

Therefore the assertion of the lemma is valid for $K = c + 2$.

Lemma 4. *Let* $\{\eta_{-k}\}_{k=1}^{\infty}$ *be a sequence of random variables,* $S_k = \eta_{-k} + \ldots + \eta_{-1}$ *,* $S_0 = 0$ *, and let* $\ldots \subseteq \mathbf{F}_{-k} \subseteq \mathbf{F}_{-k+1} \subseteq \ldots \subseteq \mathbf{F}_{-1}$ *be an increasing sequence of* $\sigma-$ *algebras, such that*

1) $\mathbf{F}_{-k} \supseteq \sigma(\eta_{-l}; l \geq k)$ for all k,

2) $\mathbf{E}(\eta_{-k} \mid \mathbf{F}_{-k-1}) \leq -\delta < 0$ *a.s.* (17)

for all k,

3) $\sup_{k \geq 1} \mathbf{E}(|\eta_{-k}| \cdot g(|\eta_{-k}|)) = c < \infty$, (18)

where g *is a function satisfying the conditions of Theorem* 5. *Then* $\overline{S} = \sup_{k \geq 0} S_k$ *is an a.e. finite random variable. Moreover, one can choose constants* K_1 *and* K_2 *so that*

$$\mathbf{P}(\overline{S} > t) \leq K_1 \cdot [g(t)]^{-1} + K_2 \cdot \int_{t\delta/2}^{\infty} [x \cdot g(x)]^{-1}\,dx .$$

R e m a r k 4. The assertion of Lemma 4 is still valid if

a) Condition (17) is replaced by the following one:

$$\mathbf{E}(\eta_{-k} \mid \mathbf{F}^{+}_{-k+1}) \leq -\delta < 0 \text{ a.s. for all } k ,$$

where $\mathbf{F}^{+}_{-k} \supseteq \sigma(\eta_{-l}; 1 \leq l \leq k)$ is some decreasing sequence of $\sigma-$ algebras; or

b) Conditions (17)-(18) are replaced by

$$\mathbf{P}(\eta_{-k} > t \mid \mathbf{F}_{-k-1}) \leq \psi(t)) \text{ a.s. for all } t, k ,$$

where $\int t\,d\psi(t) > 0$.

Proof of Lemma 4. Without loss of generality, one can assume that $\mathbf{F}_{-k} = \sigma(\eta_{-l}\,;\, l \ge k)$. Introduce the random variables $z_k = \eta_{-k} - \mathbf{E}(\eta_{-k} \mid \mathbf{F}_{-k-1})$, $Y_{k,n} = z_{n+1} + \ldots + z_{n+k}$, $\overline{Y}_{k,n} = \max(0, \max_{1 \le l \le n} Y_{l,n})$, $Y_k = Y_{k,0}$; $\overline{Y}_k = \overline{Y}_{k,0}$. Denote $d_{k,n} = \mathbf{E}(\overline{Y}_{k,n} \cdot g(\overline{Y}_{k,n}))$. Note that the sequence $\{Y_k\}$ forms a martingale and $\mathbf{E}\{|z_k| \cdot g(|z_k|)\} \le c' < \infty$ }. Besides, a.s. $z_k \ge \eta_{-k} + \delta$. Let us apply Lemma 3:

$$d_{k,n} = \mathbf{E}\{\max(0, z_{n+1} + \overline{Y}_{k-1,n+1}) \cdot g(\max(0, z_{n+1} + \overline{Y}_{k-1,n+1}))\} \le$$

$$\le \mathbf{E}\{|z_{n+1} + \overline{Y}_{k-1,n+1}| \cdot g(|z_{n+1} + \overline{Y}_{k-1,n+1}|)\} \le$$

$$\le d_{k-1,n+1} + K \cdot \mathbf{E}\{|z_{n+1}| \cdot g(|z_{n+1}|)\} + \mathbf{E}\{g(\overline{Y}_{k-1,n+1}) +$$

$$+ \overline{Y}_{k-1,n+1} \cdot g'(\overline{Y}_{k-1,n+1})) \cdot \mathbf{E}\{z_{n+1} \mid \mathbf{F}_{-n-2}\} \le d_{k-1,n+1} + K \cdot c' .$$

So, since $d_{1,l} \le c' < \infty$ for any l, we obtain by induction the estimate $d_{k,n} \le k \cdot K c' \equiv \equiv k \cdot K_1$ for all $n, k \ge 0$. Further, for any integer $t > 0$

$$\mathbf{P}(\overline{S} \ge t) \le \mathbf{P}(\overline{S}_t \ge t) + \sum_{i=1}^{\infty} \mathbf{P}\Big(\max_{t \cdot 2^{i-1} < k \le t \cdot 2^i} S_k \ge t\Big) \le$$

$$\le \mathbf{P}(\overline{Y}_t \ge t) + \sum_{i=1}^{\infty} \mathbf{P}\Big(\max_{t \cdot 2^{i-1} < k \le t \cdot 2^i} Y_k \ge t(1 + 2^{i-1}\delta)\Big) \le$$

$$\le \mathbf{P}(\overline{Y}_t \ge t) + \sum_{i=1}^{\infty} \mathbf{P}(\overline{Y}_{t \cdot 2^i} \ge t(1 + 2^{i-1}\delta)) .$$

By the Chebyshev inequality, the latter sum does not exceed the expression (here $d_k \equiv d_{k,0}$):

$$\frac{d_t}{t \cdot g(t)} + \sum_{i=1}^{\infty} \frac{d_t \cdot 2^i}{t \cdot 2^{i-1} \cdot \delta \cdot g(t \cdot 2^{i-1} \cdot \delta)} \le$$

$$\le \frac{K_1}{g(t)} + \sum_{i=1}^{\infty} \frac{K_1 \cdot t \cdot 2^i}{t \cdot 2^{i-1} \cdot \delta \cdot g(t \cdot 2^{i-1} \cdot \delta)} =$$

$$\frac{K_1}{g(t)} + \sum_{i=1}^{\infty} \frac{4K_1}{\delta} \int_{t \cdot 2^{i-2}\delta}^{t \cdot 2^{i-1}\delta} [t \cdot 2^{i-1}\delta \cdot g(t \cdot 2^{i-1}\delta)]^{-1} dx \le$$

$$\le \frac{K_1}{g(t)} + \frac{4K_1}{\delta} \sum_{i=1}^{\infty} \int_{t \cdot 2^{i-2}\delta}^{t \cdot 2^{i-1}\delta} [x \cdot g(x)]^{-1} dx = \frac{K_1}{g(t)} + \frac{4K_1}{\delta} \int_{t\delta/2}^{\infty} [x \cdot g(x)]^{-1} dx .$$

The lemma is proved.
We proceed to the

Proof of Theorem 5. We introduce the random variables $\psi_k^* = \psi_k + \varepsilon/2$ and $\zeta_k^* = \zeta_k - \varepsilon/2$, where $\varepsilon = -\mathbf{E}\psi_k$. Then

$$X(n+1) \le C_1 + C_2 + \sum_{k=\mu_{n+1}}^{n} \psi_k^* + \sum_{k=\mu_{n+1}}^{n} \zeta_k^* \equiv C_3 + \sum\nolimits_1 + \sum\nolimits_2 ,$$

where $\mu_n = \max\{ k \le n : X(k) \le C_2 \}$ and $\mu_n = 0$ if $X(k) > C_2$ for any $k \le n$. Note that

$$\sum\nolimits_1 \le \max\left(0, \sup_{j \ge 0} \sum_{k=n-j}^{n} \psi_k^*\right) \equiv \Psi^{-n+1} < \infty \text{ a.s.,}$$

where the sequence $\{ \Psi^n \}$ is stationary.

Set $\zeta_k^* = -\varepsilon/2$ for $k \le 0$ and $\eta_l = \zeta_{l+n-1}^*$. Then $\sum_2 \le \max\left(0, \sup_{l \ge 1} \sum_{k=-l}^{-1} \eta_k\right)$.

Lemma 4 implies that

$$\mathbf{P}\left(\sum\nolimits_2 > t\right) \le K_1 [g(t)]^{-1} + K_2 \int_{t\varepsilon/4}^{\infty} [x g(x)]^{-1} dx ,$$

where the right-hand side of the inequality does not depend on n. Therefore

$$\mathbf{P}(X(n+1) > t + C_3) \le \mathbf{P}(\Psi^{n+1} > t/2) + \mathbf{P}\left(\sum\nolimits_2 > t/2\right),$$

and the right-hand side goes to zero as $t \to \infty$ uniformly in n. Theorem 5 is proved.

5. On other conditions implying existence of *V*-inducing events and boundedness in probability

The essence of Sections 2-4 is, roughly speaking, as follows: having taken (7)-(8) for initial conditions, we have considered some different ways of extending them, which suffice for the construction of V-inducing events or for boundedness in probability of the sequence under study, $\{X(n)\}$. The approaches introduced above can also be applied to models with different "initial" characteristics.

Consider, for instance, a sequence $\{X(n)\}$ with values in $\mathbf{X} = \mathbf{R}_+ = [0, \infty)$, for which the inequalities

$$X(n+1) \le \alpha_n \cdot X(n) + \begin{cases} \beta_n & \text{for } X(n) > C_2, \\ C_1 & \text{for } X(n) \le C_2, \end{cases} \tag{19}$$

hold a.s., where $\{\alpha_n, \beta_n\}$ is a stationary metrically transitive sequence, $\mathbf{P}(\alpha_n > 0) = 1$, $\mathbf{E}(\ln \alpha_n)^+ < \infty$, and $\mathbf{E}(\ln \beta_n^+)^+ < \infty$.

The asymptotic properties of sequences of the form $X(n+1) = \alpha_n X(n) + \beta_n$ and those of the related processes in continuous time were studied in [24], [25].

Denote $\sigma_n = \ln \alpha_n$.

Theorem 6. *If* $\mathbf{E}\sigma_n < 0$ *or* $\sigma_n \equiv 0$ *and* $\mathbf{E}\beta_n < 0$, *then a stationary majorant can be constructed for the sequence* $\{X(n)\}$.

The last two relations $\sigma_n \equiv 0$, $\mathbf{E}\beta_n < 0$ signify the realization of Conditions (7)-(8).

Proof. For the sake of simplicity, assume that $X(0) = const \le C_1$. Let us introduce, as above, the random variable $\mu_{n+1} = \max\{k \le n+1 : X(k) \le C_2\}$. Then

$$X(n+1) \le C_2 \cdot \prod_{i=\mu_{n+1}}^{n} \alpha_i + \sum_{i=\mu_{n+1}}^{n} \beta_i \prod_{j=i}^{n-1} \alpha_j + C_1 =$$

$$= C_2 \cdot \exp\{\sum_{i=\mu_{n+1}}^{n} \sigma_i\} + \sum_{i=\mu_{n+1}}^{n} \beta_i \exp\{\sum_{j=i}^{n-1} \sigma_j\} + C_1 \le$$

$$\le \sup_{-\infty < m \le n+1} \{C_1 + C_2 \cdot \exp(\sum_{i=m}^{n} \sigma_i) + \sum_{i=m}^{n} \beta_i \cdot \exp(\sum_{j=i}^{n-1} \sigma_j)\} \equiv Y^{n+1}.$$

Let us show that $Y^{n+1} < \infty$ a.s. Indeed, for $n = -1$ the random variable $\sup_{m \le 0} (\sum_{i=m}^{-1} \sigma_i)$ is a.s. finite and

$$\sup_{m \le 0} \sum_{i=m}^{-1} \beta_i \exp(\sum_{j=i}^{-2} \sigma_j) \le \sup_{m \le -N} (\dots) + \max_{-N \le m \le 0} (\dots),$$

and the first term in the right-hand side admits the estimate

$$\mathbf{P}(\sup_{m \le -N} (\dots) > t) \le \mathbf{P}(\sup_{m \le -N} \sum_{i=m}^{-1} \beta_i^+ \cdot e^{-i(\varepsilon - \delta)} > t/2) +$$

$$+ \mathbf{P}(\sup_{m \le -N} (\sum_{i=m}^{-2} \sigma_i - (i-1)(\varepsilon - \delta) \ge 0),$$

where $\varepsilon = -\mathbf{E}\sigma_1 > 0$ and $0 < \delta < \varepsilon$ is any number. The latter inequalities imply that Y^0 is a.s. finite, and thus lead to the proof of the theorem.

Condition (19) may be generalized via approaches introduced in Sections 2 - 4.

The assertion of Theorem 6 was initiated by discussing the problems highlighted in this chapter with Prof. F.Baccelli.

CHAPTER 6. STABILITY OF STATIONARY DISTRIBUTIONS FOR RECURSIVE CHAINS

Let us consider a sequence of RC $\{{}^rX = ({}^rX(n)) \equiv ({}^rX({}^rx_0, n))\}$, $r = 1, 2, \dots, \infty$, with drivers $\{{}^r\xi_n\}$ and transition functions ${}^r\mathbf{P}(x, y, B)$ depending on the parameter *r*. All the RC ${}^rX$ assume values in a space $(\mathbf{X}, \mathbf{B_X})$ and the drivers (which are stationary metrically transitive sequences) assume values in a space $(\mathbf{Y}, \mathbf{B_Y})$. All the notations introduced above will be supplied with the left superscript *r* when they refer to the process ${}^rX$. It will be convenient to abbreviate $X = {}^\infty X$.

We shall assume the following condition to be fulfilled:

(S1) *For any* $r = 1, 2, \dots, \infty$ *the RC* $\{{}^rX(n)\}$ *satisfies Conditions (I RC) - (II RC) so that* $\{{}^rX(n)\}$ *sc-converges to a stationary sequence* $\{{}^rX^n\}$.

Moreover, in order to avoid too cumbersome notations and statements, we shall assume in addition that

(S2) *The functions* φ *and* p *involved in Condition (II RC) do not depend on their arguments* $(y_0, \dots, y_m)$ *(i.e.,* $p \equiv$ *const and* $\varphi(\dots; B) = \varphi(B)$*); the parameters* $m \geq 0$, p, *and the measure* φ *do not depend on* r.

The stability problem is to reveal conditions which provide for convergence of the stationary and non-stationary distributions of RC ${}^rX$ to the stationary ones of RC $X = {}^\infty X$.

To state the main result we need some additional notations.

Let a stationary sequence of renovating events $A_n \in \mathbf{F}^\xi_{n+m}$, $\mathbf{P}(A_n) > 0$ be given. Introduce the random variables

$$\mu_0 = \max\{k \leq -m : I(A_k) = 1\};$$

$$\mu_{j+1} = \min\{k \geq \mu_j + m : I(A_k) = 1\} \text{ for } j \geq 0;$$

$$\mu_{j-1} = \max\{k < \mu_j - m : I(A_k) = 1\} \text{ for } j \leq 0.$$

In other words, we have introduced the consecutive realization times of events A_n, separated by time intervals of length m.

For given numbers $n \geq 0$, $k_1 \leq 0 \leq k_2$, and a sequence $\{l_j, k_1 \leq j \leq k_2\}$ such that $l_{k_2} < n - m$, $l_0 = -m$, and $l_j - l_{j-1} > m$ for $j = k_1 + 1, k_1 + 2, \dots, k_2$, denote by $D_n = D_n\{k_1, k_2, \{l_j\}\}$ the event of the form

$$D_n = \bigcap_{j=k_1}^{k_2} \{\mu_j = l_j\} \cap \{\mu_{k_2+1} \geq n - m\}.$$

In particular, for $m = 0$ the occurrence of the event D_n signifies that exactly k_2 events A_j (called "successes") happen on the time interval $[1, n-1]$ (i.e., there are k_2 successes, of which the first happens at time l_1, the second one at time $l_2, \dots,$ and k_2-th happens at time l_{k_2}; the $(k_2 + 1)$-th success may happen at time $n - m$ or later). The consecutive "successes" on the negative semi-axis happen at times l_j, $k_1 \leq j \leq 0$, respectively. A similar, but more complicated, verbal description may be provided also for $m > 0$.

For $x \in \mathbf{X}$, $B \in \mathbf{B}_\mathbf{X}$, and event $D \in \mathbf{F}^\xi_n$, denote by $\mathbf{P}_{(n)}(x, B, D)$, $n \geq 0$, the probability

$$\mathbf{P}_{(n)}(x, B, D) = \mathbf{P}(\{X(x, n+1) \in B\} \cap D).$$

Theorem 1. *Assume that the following conditions are valid along with* (S1), (S2):

(S3) $\mathbf{P}(\bigcup_{i=1}^{n} {}^rA_i) \to 1$ *for* $n \to \infty$ *uniformly over r, i.e.,*

$$\liminf_{r \to \infty} \mathbf{P}(\bigcup_{i=1}^{n} {}^rA_i) \equiv d_n \to 1 \textit{ for } n \to \infty;$$

(S4) $\int \varphi(dx)\ {}^r\mathbf{P}_{(n)}(x, B, {}^rD_n) \to \int \varphi(dx)\,\mathbf{P}_{(n)}(x, B, D_n)$

as $r\to\infty$ *for some* $B \in \mathbf{B}_{\mathbf{X}}$ *and any* $n, k_1, k_2, \{l_j\}$ *satisfying the conditions listed in the definition of the event* D_n. *Then*

$$\mathbf{P}({}^rX(n) \in B) \to \mathbf{P}(X^0 \in B) \text{ as } n, r \to\infty$$

for this B. In particular, $\mathbf{P}({}^rX^0 \in B) \to \mathbf{P}(X^0 \in B)$.

Remark 1. One may introduce a series of intrinsic and comparatively simple conditions which are sufficient for (S3) - (S4). Assume that the spaces $\mathbf{X}$ and $\mathbf{Y}$ are provided with the weak convergence topologies associated with the corresponding $\sigma-$ algebras. Suppose, for the sake of simplicity, that ${}^rX$ are SRS with drivers $\{{}^r\xi_n\}$ for all r: ${}^rX(n+1) = {}^rf({}^rX(n), {}^r\xi_n)$; the events ${}^rA_n \in {}^r\mathbf{F}_{n+m}$ are renovating for ${}^rX(n)$, i.e., ${}^rX(n+m+1) = {}^rg({}^r\xi_n, \dots, {}^r\xi_{n+m})$ a.s. on ${}^rA_n$; and ${}^rA_n$ can be represented in the form ${}^rA_n = \{{}^rh_n < C\}$, where $C = const$, ${}^rh_n = h({}^r\xi_{n+m}, {}^r\xi_{n+m-1}, \dots)$, while $h : \mathbf{Y}^\infty \to \mathbf{R}$ is some measurable function.

Then (see [22]) Conditions (S3) - (S4) will be satisfied if

1) the finite-dimensional joint distributions of $({}^r\xi_n, {}^rh_n)$ converge weakly as $r\to\infty$ to the distributions of (ξ_n, h_n) and $\mathbf{P}(h_n = C) = 0$;

2) the functions $g_k(x_0, \dots, x_k)$ defined by the equalities $g_m = g$ and $g_{k+1}(x_0, \dots, x_{k+1}) = f(g_k(x_0, \dots, x_k), x_{k+1})$ are a.s. continuous with respect to the distribution of $(\xi_0, \dots, \xi_k)$ for all k, and

$$\sup_{(x_0,\dots,x_k)\in B_k^r} |\,{}^rg_k(x_0, \dots, x_k) - g_k(x_0, \dots, x_k)\,| \to 0$$

for any sequence of sets $\{B_k^r;\, r \ge 1\}$ such that $\mathbf{P}(({}^r\xi_0, \dots, {}^r\xi_k) \notin B_k^r) \to 0$ for $r\to\infty$;

3) the set $B \in \mathbf{B}_{\mathbf{X}}$ is such that $\mathbf{P}(X^0 \in \partial B) = 0$, where ∂B denotes the boundary of the set B .

Proof of Theorem 1. As before, we shall restrict ourselves to the case $m = 0$. Consider all the random variables as defined on the same probability space.

As in the proof of Theorem 4.3, we consider the "extended" RC ${}^r\tilde{X}(n) = ({}^rX(n), \delta_n)$ with the drivers ${}^r\tilde{\xi}_n = ({}^r\xi_n, \delta_n)$ and introduce the events ${}^rC_n = {}^rA_n \cap \{\delta_n = 1\}$. Note that Condition (S3) implies

$$\liminf_{r\to\infty} \mathbf{P}\Big(\bigcup_{i=1}^{n} {}^rC_i\Big) \to 1 \tag{1}$$

for $n\to\infty$.

Indeed, consider the event

$$E_{lk} = \{\delta_l = \delta_{l+1} = \dots = \delta_{l+k} = 1\}.$$

Then for $k \le n$

$$\liminf_{r\to\infty} \mathbf{P}\Big(\bigcup_{i=1}^{n} {}^rC_i\Big) \geq \sum_{l=0}^{n-k} \mathbf{P}\Big(E_{lk} \cap \Big(\bigcap_{j=0}^{l-1} \overline{E}_{jk}\Big)\Big) \cdot \liminf_{r\to\infty} \mathbf{P}\Big(\bigcup_{i=l}^{l+k} {}^rA_i\Big) =$$

$$= \mathbf{P}\Big(\bigcup_{l=0}^{n-k} E_{lk}\Big) \cdot \liminf_{r\to\infty} \mathbf{P}\Big(\bigcup_{i=0}^{k} {}^rA_i\Big) = \mathbf{P}\Big(\bigcup_{l=0}^{n-k} E_{lk}\Big) \cdot d_k \ .$$

Let us choose k so that $d_k \geq 1-\varepsilon$, and define n so that $\mathbf{P}\Big(\bigcup_{l=0}^{n-k} E_{lk}\Big) \geq 1-\varepsilon$. Then the limit we look for will be no less than $1-2\varepsilon$. The choice of $\varepsilon > 0$ being arbitrary, (1) is proved.

Denote then ${}^r\gamma_n = \max\{k \leq n : I({}^rC_k) = 1\}$. Since

$$\mathbf{P}({}^r\gamma_n < n-j) = \mathbf{P}\Big(\bigcap_{i=1}^{j} {}^r\overline{C}_{n-j}\Big) \ ,$$

there exists for any $\varepsilon > 0$ a number L such that

$$\mathbf{P}({}^r\gamma_n < n-L) \leq \varepsilon \quad \text{for all } r = 1, 2, \dots, \infty \ .$$

Thus for $n >> L$

$$\mathbf{P}({}^r\tilde{X}(n+1) \in B) = \sum_{i=m}^{L} \mathbf{P}({}^r\tilde{X}(n+1) \in B ; {}^r\gamma_n = n-i) + O(\varepsilon) \ .$$

Let us demonstrate that for all i

$$\mathbf{P}({}^r\tilde{X}(n+1) \in B ; {}^r\gamma_n = n-i) \to \mathbf{P}(\tilde{X}(n+1) \in B ; \gamma_n = n-i) \text{ as } r \to \infty \ .$$

Indeed, the relations

$$\mathbf{P}({}^r\tilde{X}(n+1) \in B ; {}^r\gamma_n = n-i) = \mathbf{P}({}^r\tilde{X}(s+1) \in B ; {}^r\gamma_s = 0) =$$

$$= \sum_{1\leq k\leq s} \ \sum_{l_1 < \dots < l_k \leq s} \mathbf{P}({}^r\tilde{X}(s+1) \in B ; {}^r\mu_0 = 0 ; {}^r\mu_j = l_j ; 1 \leq j \leq k ;$$

$${}^r\mu_{k+1} > s ; \delta_0 = 1 ; \delta_{l_j} = 0 ; 1 \leq j \leq k)$$

hold for $s = n-i$. Each component of the latter sum can be represented as

$$\mathbf{P}({}^r\tilde{X}(s+1) \in B ; {}^r\mu_0 = 0 ; \delta_0 = 1 ; {}^r\mu_j = l_j ; 1 \leq j \leq k ; {}^r\mu_{k+1} > s) -$$

$$- \sum_{t=1}^{k} \mathbf{P}({}^r\tilde{X}(s+1) \in B ; {}^r\mu_0 = 0 ; \delta_0 = 1 ; {}^r\mu_j = l_j ; 1 \leq j \leq k ;$$

$${}^r\mu_{k+1} > s ; \delta_{l_j} = 0 ; 1 \leq j \leq k ; \delta_{l_t} = 1) \ .$$

The minuend of the latter equals

$$p \cdot \int \varphi(dx)\, \mathbf{P}({}^r\tilde{X}(x, s) \in B ; {}^r\mu_0 = -1 ; {}^r\mu_j = l_{j-1} ; 1 \leq j \leq k ; {}^r\mu_{k+1} > s-1) \ , \qquad (2)$$

and each of the subtrahends coincides, naturally, with

$$p^2 \cdot (1-p)^{t-1} \int \varphi(dx) \cdot \mathbf{P}({}^r\tilde{X}(x, s-l_t-1) \in B;$$

$${}^r\mu_{j-t} = l_j - l_t - 1;\ 1 \le j \le k;\ {}^r\mu_{k+1-t} > s - l_t - 1). \tag{3}$$

By virtue of Condition (S4) each of expressions (2) and (3) converges, for $r \to \infty$, to the corresponding expression with superscript r omitted. Thus the theorem is proved.

Other approaches to the stability problems are considered in [26] - [27].

CHAPTER 7. ERGODICITY OF THE PROCESSES ADMITTING EMBEDDED RECURSIVE CHAINS

1. The main definitions

Let $Z = \{Z(t) = Z(t,x), t \in \mathbf{T}\}$, $Z(x,0) = x$ be arbitrary $\mathbf{X}$-valued random processes. Time $t \ge 0$ may be either discrete ($\mathbf{T} = \{0,1,2,\dots\}$), or continuous ($\mathbf{T} = [0,\infty)$).

One of the natural approaches to the study of ergodicity conditions for the process Z is connected with construction of the so-called "embedded" sequences, for which ergodicity can be established. One usually calls a sequence embedded if it is constituted by the values of the process at some "embedded" (usually, Markov) times. Let

$$0 \le T_0 < T_1 < \dots < T_n < \dots;\ T_n \in \mathbf{T};\ T_n \to \infty \text{ a.s. for } n \to \infty \tag{1}$$

be some random sequence. It is natural to expect the ergodicity of the process Z to follow from the ergodicity of the sequence $X(n) \equiv Z(T_n)$ under fairly general assumptions (see [28] for another approach to ergodicity studies of the processes in continuous time).

We shall assume T_n to be stopping times, i.e., for any n, t the event $\{T_n \le t\}$ belongs to the $\sigma-$algebra $\mathbf{F}_{(t)} = \sigma\{Z(u); u \le t\}$. Denote $\nu_t = \max\{k : T_k \le t\}$, so that the random variables ν_t and T_{ν_t} are measurable with respect to the $\sigma-$algebra $\mathbf{F}_{(t)}$. Define the $\sigma-$algebra $\mathbf{F}^*_{(t)}$, generated by sets of the form $B \cap \{T_{\nu_t} \ge u\}$, $B \in \mathbf{F}_{(u)}$, $u \le t$, $u \in \mathbf{T}$. It is clear that $\mathbf{F}^*_{(t)} \subseteq \mathbf{F}_{(t)}$.

Definition 1. We shall say that the process Z admits an embedded RC if there exist a sequence of stopping times $\{T_n; n \ge 0\}$ satisfying (1), a measurable space $(\mathbf{Y}, \mathbf{B_Y})$, and a sequence of $\mathbf{Y}$-valued random variables η_n, which are measurable with respect to the $\sigma-$algebras $\mathbf{F}_{(T_n)}$, such that

1) the sequence $\xi_n \equiv (e_n \equiv T_{n+1} - T_n, \eta_n), n \ge 0$ is stationary;

2) the sequence $X(n) = Z(T_n), n \ge 0$ forms a RC with the driver $\{\xi_n\}$;

3) for any t the conditional distribution (with respect to $\mathbf{F}^*_{(t)}$) of the random variable $Z(t)$ depends on $X(\nu_t)$ and $t - T_{\nu_t}$ only, i.e.,

$$\mathbf{P}(Z(t) \in B \mid \mathbf{F}^*_{(t)}) = \mathbf{P}(Z(t) \in B \mid X(\nu_t); t - T_{\nu_t}) \equiv G(X(\nu_t), t - T_{\nu_t}, B) \text{ a.s.}$$

for all $B \in \mathbf{B_X}$, where the function $G(x,u,B)$ is measurable with respect to the pair of variables (x,u) for any $B \in \mathbf{B_X}$ and is a probability measure on $\mathbf{X}$ for any (x,u).

R e m a r k 1 . One may define processes admitting embedded RC in another way. Namely, if we introduce σ– algebras $\mathbf{F}^{**}_{(t)}$ generated by sets of the form $B \cap \{T_{\nu_t+1} \geq u\}$, $B \in \mathbf{F}_{(u)}$, $u \in \mathbf{T}$, and replace Condition 3) in Definition 1 by the following one,

3') the conditional distribution of $Z(t)$ with respect to $\mathbf{F}^{**}_{(t)}$ depends on $X(\nu_t)$, $X(\nu_t+1)$, ξ_{ν_t+1}, and $t - T_{\nu_t}$ only,

then we obtain a different version of "embedding". The statements below can be reformulated and proved for the processes admitting "embedding" in the sense of 3').

Let us obtain first the ergodicity theorems for processes admitting embedded RC, and then clarify by examples, what assumptions in terms of the process Z provide for the existence of an embedded RC. In case of continuous time we shall require the trajectories of the process Z to be continuous and assume that *the spaces* $\mathbf{X}, \mathbf{Y}$ *are metric.*

Let us start from one simple case.

2. Ergodicity of processes in the case when the driver $\{\xi_n\}$ of the embedded chain consists of independent elements

Note that, in the conditions of this section, the embedded RC is a MC. Consider the case of *continuous* time.

Theorem 1. *Assume that the process Z admits an embedded chain, $T_0 < \infty$ a.s., the RC $\{X(n)\}$ sc-converges to some stationary RC $\{X^n\}$, and the sequence $\{\xi_n\}$ consists of i.i.d. random elements. Let also*

1) *the random variable e_0 be non-lattice,* $\mathbf{E}\, e_0 < \infty$,

2) *the trajectories of the process Z be right (or left) continuous with probability one (this condition may be replaced by the following one: a.e. in the path space of the process Z the closure of the set of discontinuity points has zero Lebesgue measure).*

Then there exists a non-singular probability distribution $\mathbf{P}$ *on* $(\mathbf{X}, \mathbf{B_X})$ *such that weak convergence*

$$\mathbf{P}_t(\cdot) \equiv \mathbf{P}(Z(t) \in \cdot) \Rightarrow \mathbf{P}(\cdot) \tag{2}$$

takes place for $t \to \infty$, or, which is equivalent,

$$\mathbf{E}\, h(Z(t)) \to \int h(x)\, \mathbf{P}(dx)$$

for any bounded continuous function h.

If we replace the assumption that e_0 is non-lattice by the stronger condition ,

3) *the distribution of the random variable e_0 has an absolutely continuous component, then Condition* 2) *becomes redundant and there is the convergence in total variation:*

$$\sup_{B \in \mathbf{B_X}} |\mathbf{P}_t(B) - \mathbf{P}(B)| \to 0 \ \textit{as}\ t \to \infty . \tag{3}$$

If Condition 3) *holds, the spaces* $\mathbf{X}$ *and* $\mathbf{Y}$ *need not be metric.*

Proof. It is sufficient to consider only the case when $\{X(n)\}$ forms a SRS. Let V and $G_n = \{X(n) \in V\} \cap A_n^{(2)}$ be the inducing set and the event from the statement of Theorem 2.9. Introduce the event $\tilde{D}_n = \{e_n \leq M, e_{n+1} \leq M, \dots, e_{n+m} \leq M\}$, where the number M is large enough for the event $D_n \equiv G_n \cap \tilde{D}_n$ to have a positive probability.

Introduce the random variables $\gamma_0 = 0$ and $\gamma_j = T_{\mu_j}$ for $j \geq 1$. Denote by $\mu_{j+1} = \min\{ n > \mu_j + m : I(G_n) = 1 \}$ the consecutive times of realization of the events G_n defined in Theorem 2.9. Set $\psi_j = \gamma_j - \gamma_{j-1}$. Theorem 2.9 implies that $\{ \psi_j \}$ is a sequence of independent and, for $j \geq 2$, identically distributed random variables; thereto ψ_j are non-lattice and, by the Wald identity, $\mathbf{E}\,\psi_2 < \infty$.

Denote by $H(y)$ the renewal function for the sequence $\{ \gamma_j \}$. Without loss of generality one may assume $T_0 = 0$. Let $h : \mathbf{X} \to \mathbf{R}$, $0 \leq h \leq 1$, be an arbitrary non-negative continuous bounded function. For $L = mM$ the process $Z(t) = Z(x,t)$ satisfies equalities:

$$\mathbf{E}\,h(Z(t+L)) = \sum_{j=0}^{\infty} \mathbf{E}(h(Z(t+L));\gamma_j \leq t \leq \gamma_{j+1}) =$$

$$= \mathbf{E}(h(Z(t+L));\gamma_1 > t) +$$

$$+ \sum_{j=1}^{\infty} \int_0^t d\,\mathbf{P}(\gamma_j < u)) \cdot \mathbf{E}(h(Z(t+L))I(\gamma_{j+1} > t) \mid \gamma_j \in du) =$$

$$= \mathbf{E}(h(Z(t+L));\gamma_1 > t) +$$

$$+ \sum_{j=1}^{\infty} \int_0^t d\,\mathbf{P}(\gamma_j < u)) \cdot \mathbf{E}(h(Z(w,t+L-u))I(\gamma_1 > t-u) \mid D_0) =$$

$$= \mathbf{E}(h(Z(t+L));\gamma_1 > t) +$$

$$+ \int_0^t dH(u) \cdot \mathbf{E}(h(Z(w,t+L-u))I(\gamma_1 > t-u) \mid D_0) ,$$

where the point $w \in V$ is arbitrary.

In the sequel we use the same argumentation as in the proof of the ergodicity theorem for regenerative processes (see, e.g., [29]). Introduce the random process

$$\varphi(u) = g(Z(w,L+u)) \cdot I(\gamma_1 > u) \cdot I(D_0) \cdot [\mathbf{P}(D_0)]^{-1} , \; u \geq 0,$$

and denote $F(u) = \mathbf{E}\,\varphi(u)$. Here

$$\mathbf{E}\,g(Z(t+L)) = \int_o^t dH(u)F(t-u) + \mathbf{E}(g(Z(t+L));\gamma_1 > t) ,$$

and to prove (2) it suffices to check whether the fact that the function $F(u)$ is directly Riemann integrable (see [30]). Condition 2) implies that trajectories of the process $\varphi(u)$ are right-continuous with probability one. Thus, by the Lebesgue majorated convergence theorem on limit transition under the integral sign, the function $F(u)$ is also right-continuous. Moreover,

$$F(u) \leq \mathbf{P}(\gamma_1 > u \mid D_0), \quad \int_0^{\infty} \mathbf{P}(\gamma_1 > u \mid D_0)\,du = \mathbf{E}(\gamma_1 \mid D_0) < \infty .$$

As demonstrated in, e.g., [29], under these conditions the function F is directly Riemann integrable and

$$\mathbf{E}\, g(Z(t+L)) \to \frac{1}{a} \int_0^{\infty} F(u)\, du .$$

Hence (2) is proved.

Let us prove the second statement. The calculations above demonstrate that for any $B \in \mathbf{B}_{\mathbf{X}}$

$$\mathbf{P}(Z(t+L) \in B) = \int_0^t dH(u) \cdot F_1(t-u) + \mathbf{P}(Z(t+L) \in B ; \gamma_1 > t) ,$$

where $F_1(u) = \mathbf{P}(Z(x_0, u+L) \in B ; \gamma_1 > u \mid D_0)$.

Let us show that

$$\mathbf{P}(Z(t+L) \in B) \to \frac{1}{a} \int_0^{\infty} F_1(u)\, du \tag{4}$$

uniformly in $B \in \mathbf{B}_{\mathbf{X}}$. Since $F_1(u) \le \mathbf{P}(\gamma_1 > u \mid D_0)$, $\mathbf{E}(\gamma_1 \mid D_0) < \infty$, in order to prove (4) it suffices to verify that

$$I_b \equiv \left| \int_0^b (-d_u H(t-u)) F_1(u) - \frac{1}{a} \int_0^b F_1(u)\, du \right| \to 0$$

uniformly in $B \in \mathbf{B}_{\mathbf{X}}$ for arbitrary fixed $b < \infty$. Set $R(u) = H(u) - u/a$. Condition 3) implies that the random variables $\{\psi_i\}$ have absolutely continuous components. Thus $\int_{t-b}^{t} |dR(u)| \to 0$ as $t \to \infty$ for any $b < \infty$ (see, e.g., [31]), . As for any $B \in \mathbf{B}_{\mathbf{X}}$

$$I_b = \left| \int_0^b F_1(u)\, dR(t-u) \right| \le \int_0^b |dR(t-u)| = \int_{t-b}^{t} |dR(u)| ,$$

the second statement of the theorem is also proved.

In the case of discrete time, the following assertion is valid.

Theorem 2. *Assume that the process Z admits an embedded RC, $T_0 < \infty$ a.e., the RC $\{X(n)\}$ sc-converges to some stationary RC $\{X^n\}$, and the sequence $\{\xi_n\}$ consists of i.i.d. random variables. If, moreover, the G.C.D. of those k , for which $\mathbf{P}(e_1 = k) > 0$, is equal to one, then there exists a non-singular probability distribution* $\mathbf{P}$ *on $(\mathbf{X}, \mathbf{B}_{\mathbf{X}})$ such that convergence* (3) *occurs.*

Proof. For $t = 1, 2, \ldots$ and a fixed integer b

$$\int_0^b (-d_u H(t-u)) F_1(u) = \sum_{k=1}^{b} (H(t-k) - H(t-k-1)) F_1(k) \to \frac{1}{a} \sum_{k=1}^{b} F_1(k),$$

where the function F_1 was introduced in the proof of Theorem 1; thereto, the convergence is uniform over $B \in \mathbf{B}_{\mathbf{X}}$. According to the remarks above, this implies the convergence in total variation. The theorem is proved.

3. Ergodicity of processes admitting embedded Markov chains

The statements of Theorems 1 and 2 stay valid in a somewhat more general situation. Let us introduce the following

Definition 2. Let us say that the process Z *admits an embedded MC*, if there exists a sequence of Markov times $\{ T_n \}$ satisfying relations (1) such that

1) the sequence $X(n) = Z(T_n)$ constitutes a homogeneous MC;

2) for any $n \geq 0, t \geq 0$ the joint distribution of $\{ Z(T_n + t), \{ e_{n+k}, k \geq 0 \} \}$ depends on $Z(T_n) = X(n)$ and t only, i.e.,

$$\mathbf{P}(Z(T_n + t) \in B, \{ e_{n+k}, k \geq 0 \} \in D \mid \mathbf{F}_{(T_n)}) =$$

$$= \mathbf{P}(Z(T_n + t) \in B, \{ e_{n+k}, k \geq 0 \} \in D \mid \sigma(X(n)))$$

a.s. for any $B \in \mathbf{B}_{\mathbf{X}}$, $D \in \mathbf{B}_{\mathbf{R}_+^\infty}$.

Note that this definition lacks the independence requirement for the elements of $\{ \xi_n \}$ (in terms of Definition 1).

If we denote $e_n = T_{n+1} - T_n$, then, evidently, the sequence $\{ X(n), e_n \}$ also forms a homogeneous MC; thereto the distribution of e_n depends on $X(n)$ only.

Note also that if the process Z admits an embedded MC, it is not necessarily a Markov process.

Apparently, for the first time the notion of an embedded MC was introduced by Kendall [32]. The up-to-date literature employs various definitions of processes admitting embedded MC (semi-Markov or, according to Asmussen, regenerative processes, etc.; see [29], [33], and the references in these books).

Suppose that a MC $X = \{ X(n) \}$ satisfies Conditions (I) - (II) (see Chapter 2). Let a number $n_1 > 0$ be such that $\mathbf{P}(\tau_V(\varphi) = n_1) \equiv q > 0$. Define a probability measure $\varphi^{(1)}$ on $\mathbf{X}$ by the relation

$$\varphi^{(1)}(B) = \mathbf{P}(X(\varphi, n_1) \in B \cap V) / q. \tag{5}$$

for $B \in \mathbf{B}_{\mathbf{X}}$. By definition, $\varphi^{(1)}(V) = 1$. Besides, for any $x \in V$, $B \subseteq V$ and $B \in \mathbf{B}_{\mathbf{X}}$,

$$\mathbf{P}(X(x, m + 1 + n_1) \in B) \geq p \cdot q \cdot \varphi^{(1)}(B).$$

Consider the MC $X^{(1)} = \{ X^{(1)}(n), n \geq 0 \}$, where $X^{(1)}(n) = X((m + 1 + n_1) \cdot n)$. It is clear that if a MC X satisfies Conditions (I) - (II), then the MC $X^{(1)}$ satisfies Conditions (I) - (III) for $p^{(1)} = p \cdot q$, $m^{(1)} = 0$, $n_1^{(1)} = 0$, and for $\varphi^{(1)}$ defined in (5), where the superscript "(1)" signifies correspondence to the chain $X^{(1)}$. Moreover, if the process Z admits the embedded MC X, then, evidently, it admits also the embedded MC $X^{(1)}$.

Therefore we may assume, without loss of generality, that the process Z admits an embedded chain X satisfying Conditions (I) - (III) for $m = 0$. It was noted above (see [7], [8] and also Chapters 2, 4) that the MC $\tilde{X} = \{ X(n), \delta_n \}$ possessing a "positive" atom can be defined on an extended probability space. If we define in addition a random variable $\delta(t)$ assuming value δ_n on the set $T_n \leq t < T_{n+1}$, then, evidently, the process $\tilde{Z}(t) = (Z(t), \delta(t))$ admits the embedded MC $\tilde{X}$.

Consider the consecutive hitting times $0 \le \tilde{\mu}_1 < \tilde{\mu}_2 < \dots$ of the "atom" $\tilde{x} = (V, 1)$ by $\tilde{X}(n)$, and assume that $\hat{T}_n = t_{\tilde{\mu}_n}$. Then it is easy to see that the process $\tilde{Z}$ admits the "trivial" embedded RC $\hat{X} = \{\hat{X}(n) = \tilde{X}(\hat{T}_n) \equiv \tilde{x}\}$ with the driver $\hat{\xi}_n = (\hat{e}_n, \cdot)$ which is a sequence of i.i.d. random variables. Here $\mathbf{E}\hat{e}_n$ is finite if

$$\sup_{x \in \mathbf{X}} \mathbf{E}\{e_0 \mid X(0) = x\} < \infty. \tag{6}$$

Denote by $\tau(y)$ a random variable with the distribution

$$\mathbf{P}(\tau(y) > t) = \mathbf{P}(T_\mu - T_0 > t \mid X(0) \in dy),$$

where μ is the first positive hitting time of the set V by $X(n)$ and φ is the measure from Condition (II).

Theorem 3. *Assume that the process* $\{Z(t), t \in [0, \infty)\}$ *admits an embedded MC* X *and the conditions below are fulfilled:*

1) *the MC* X *satisfies Conditions* (I) - (III) *;*
2) (6) *takes place;*
3) *the distribution*

$$\mathbf{P}(\hat{\tau} > t) = \int \varphi(dy) \cdot \mathbf{P}(\tau(y) > t)$$

of the random variable $\hat{\tau}$ *is a non-lattice one;*

4) *Condition* 2) *of Theorem* 1 *holds.*

Then the distributions $\mathbf{P}_t(\cdot) = \mathbf{P}(Z(t) \in \cdot)$ *weakly converge to some probability distribution. If we require instead of* 3) *the distribution of* $\hat{\tau}$ *to possess an absolutely continuous component, Condition* 4) *becomes redundant and the convergence is in total variation.*

Theorem 3 follows immediately from Theorem 1 and the argumentation above. Indeed, the process Z admits the trivial embedded RC $\hat{X}$, and Conditions 2), 3) of Theorem 3 imply Condition 1) of Theorem 1.

An analogous assertion may be stated for the discrete case.

4. Ergodicity of processes in the case of embedded RC with stationary drivers

Suppose that a process Z admits an embedded RC X with the driver $\xi_n = (e_n, \eta_n)$, which is a stationary metrically transitive sequence. As remarked above, without loss of generality one may assume X to be a SRS with driver ξ_n. Assume in addition that the following condition is satisfied:

(A) (T_n, η_n) is a stationary marked point process (SMPP) (viz., (T) is a point process and (η_n) is a sequence of corresponding stationary "marks").

Definition of SMPP can be encountered, for instance, in [3], [5], [25].

Theorem 4. *If Condition* (A) *is fulfilled and there exists a stationary sequence of "positive" renovating events* $A_n \in \mathbf{F}^{\xi}_{n+m}$ *for SRS* X*, then there exists a probability measure* $\mathbf{P}(\cdot)$ *on* $(\mathbf{X}, \mathbf{B_X})$*, such that the convergence in total variation*

$$\sup_{B \in \mathbf{B_X}} |\mathbf{P}(Z(t) \in B) - \mathbf{P}(B)| \to 0 \quad \textit{as } t \to \infty$$

takes place.

Proof. Define the random variables $\beta_t = t - T_{\nu_t}$;

$$\alpha(n) = \min\{k > m : I(A_{n-k}) = 1\}; \ \alpha_t = \alpha(\nu_t).$$

Further, set $l \geq m$

$$\psi_{n,l} = f^{l-m}(g(\xi_n, \dots, \xi_{n+m}), \xi_{n+m+1}, \dots, \xi_{n+l}),$$

where, as before, f^i are iterations of the function f and g is the function involved in the definition of renovating events. Finally, put $\psi(t) = \psi_{\nu_t - \alpha_t, \alpha_t - 1}$.

Condition (A) stipulates that the process $(\eta_{\nu_t}, \alpha_t, \beta_t)$ is stationary; hence, the process $(\psi(t), \beta_t)$ also obtains this property.

Define the measure $\mathbf{P}(\cdot)$ by the equalities $\mathbf{P}(B) = \mathbf{E}\,G(\psi(t), \beta_t, B)$, where $t > 0$ is arbitrary. Let $\{X^n\}$ be the stationary sequence, to which the sequence $\{X(n)\}$ sc-converges. Note that

$$X^{\nu_t} = \psi(t) \text{ a.s. for all } t$$

and

$$\mathbf{P}(X(\nu_t) = X^{\nu_t}) \geq \mathbf{P}(\alpha_t \leq \nu_t) \to 1 \text{ for } t \to \infty.$$

Thus

$$\mathbf{P}(Z(t) \in B) = \mathbf{E}\,G(X(\nu_t), \beta_t, B) =$$

$$= \mathbf{E}\,G(\psi(t), \beta_t, B) + O(\mathbf{P}(\alpha_t > \nu_t)),$$

and this probability converges to $\mathbf{P}(B)$ as $t \to \infty$ uniformly in $B \in \mathbf{B}_{\mathbf{X}}$. The theorem is proved.

The case, when Condition (A) fails and the sequence (T_n, η_n) converges in a certain sense only to some SMPP, is technically more sophisticated, so we found it expedient not to consider this case in the present paper.

5. Examples of processes admitting embedded RC

Let us consider one particular case which is rather important for applications, when a process in continuous time is determined by the embedded SRS. Such processes are studied, for instance, in queueing models.

Suppose that we have specified a SRS $\{X(n+1) = f(X(n), \xi_n)\}$, and the space, where the elements of the driver ξ_n assume their values, is $[0, \infty) \times \mathbf{Y}$ (with $(\mathbf{Y}, \mathbf{B_Y})$ some measurable space). In this case it is natural to write down the random variables ξ_n in the form $\xi_n = (e_n, \eta_n)$, where $e_n \geq 0$ a.s.. Assume additionally that $e_n > 0$ a.s. Denote $T_0 = 0$ and $T_n = e_0 + \dots + e_{n-1}$ for $n \geq 1$.

Let $h : \mathbf{X} \times [0, \infty) \to \mathbf{X}$ be a measurable function such that $h(x, 0) = x$ for all $x \in \mathbf{X}$. Define the process $Z(t)$ according to the rule:

$$Z(t) = h(X(n), t - T_n) \tag{7}$$

for $T_n \leq t < T_{n+1}$, $n = 0, 1, \dots$. The definition implies, in particular, the equality $Z(T_n) = X(n)$ a.s.

It is easy to see that the process $Z(t)$ defined by (7) admits the embedded RC X. Ergodicity conditions for the processes of the form (7) were considered in [25], [34].

The processes of virtual waiting times, studied in queueing theory, may be considered as examples of the processes of the form (7). In particular, for systems $G/G/1$ the virtual waiting time is defined by the equalities $Z(t) = (X(\nu_t) - (t - T_{\nu_t}))$, $t \geq 0$, where $T_n = e_0 + \ldots + e_{n-1}$ is the arrival time of the n-th customer, s_n is its service time, and $X(n) = (X(n-1) - e_{n-1})^{+} + s_n$ is the sojourn time of the n-th customer in the system.

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