Tail Asymptotics for the Supremum of a Random Walk when the Mean Is Not Finite[1]

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Abstract. We consider the sums $S_n = \xi_1 + \cdots + \xi_n$ of independent identically distributed random variables. We do not assume that the ξ's have a finite mean. Under subexponential type conditions on distribution of the summands, we find the asymptotics of the probability $\mathbf{P}\{M > x\}$ as $x \to \infty$, provided that $M = \sup\{S_n,\ n \ge 1\}$ is a proper random variable. Special attention is paid to the case of tails which are regularly varying at infinity.

We provide some sufficient conditions for the integrated weighted tail distribution to be subexponential. We supplement these conditions by a number of examples which cover both the infinite- and the finite-mean cases. In particular, we show that the subexponentiality of distribution F does not imply the subexponentiality of its integrated tail distribution F^I.

Keywords: supremum of sums of random variables, large deviation probabilities, subexponential distribution, integrated weighted tail distribution

1. Introduction

Let ξ, ξ_1, ξ_2, ... be independent random variables with common non-degenerate distribution F on the real line $\mathbf{R}$. We let $F(x) = F((-\infty, x])$ and $\overline{F}(x) = 1-F(x)$. In general, for any distribution G, we denote its tail by $\overline{G}(x) = G((x,\infty))$. In this paper, an important role is played by *the negative truncated mean function*

$$m(x) \equiv \mathbf{E}\min\{\xi^-, x\} = \int_0^x \mathbf{P}\{\xi^- > y\}\, dy, \quad x \ge 0,$$

where $\xi^- = \max\{-\xi, 0\}$; the function $m(x)$ is continuous, $m(0) = 0$ and $m(x) > 0$ for any $x > 0$.

Put $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$, and

$$M = \sup\ \{S_n,\ n \ge 0\}.$$

[1]Supported by EPSRC grant No. R58765/01, INTAS grant No. 265 and RFBR grant No. 02-01-00358

Our main assumption is that M is finite a.s. The latter occurs if and only if $S_n \to -\infty$ as $n \to \infty$ with probability one (see Theorem 1 in [13, Chapter XII, Section 2]). It is known that

(i) if $\mathbf{E}|\xi| < \infty$, then $S_n \to -\infty$ a.s. as $n \to \infty$ if and only if $\mathbf{E}\xi < 0$;

(ii) if $\mathbf{E}|\xi| = \infty$, then $S_n \to -\infty$ a.s. as $n \to \infty$ if and only if

$$\int_0^\infty \frac{x}{m(x)} F(dx) \ \text{ is finite,} \tag{1}$$

see Corollary 1 in [12]. Note that the function $\frac{x}{m(x)}$ is increasing, since

$$\frac{d}{dx}\frac{x}{m(x)} = \frac{m(x) - xm'(x)}{m^2(x)} = \frac{m(x) - x\mathbf{P}\{\xi^- > x\}}{m^2(x)} \geq 0. \tag{2}$$

In the case (ii), $m(x) \to \infty$ as $x \to \infty$, with necessity. Roughly speaking, the condition (1) means that the right tail of the distribution F is lighter than the left one.

The main goal of the present paper is to investigate the asymptotic behaviour of the probability $\mathbf{P}\{M > x\}$ as $x \to \infty$ when the distribution of the summands is heavy-tailed. As far as applications are concerned, (a) in queueing, M coincides in distribution with the stationary waiting time in the corresponding $GI/G/1$ queue; (b) in risk theory, $\mathbf{P}\{M > x\}$ is the probability of ruin.

We recall the definitions of some classes of functions and distributions which will be used in the sequel.

Definition 1. The function f is called *long-tailed* if, for any fixed t, the limit of the ratio $f(x+t)/f(x)$ is equal to 1 as $x \to \infty$. We say that the distribution G is long-tailed (and write $G \in \mathscr{L}$) if the function $\overline{G}(x)$ is long-tailed.

Definition 2. The distribution G on $\mathbf{R}^+$ with unbounded support belongs to the class $\mathscr{S}$ (and is called *a subexponential distribution*) if the convolution tail $\overline{G*G}(x)$ is asymptotically equivalent to $2\overline{G}(x)$ as $x \to \infty$.

It is shown in [6] that any subexponential distribution G is long-tailed with necessity. Sufficient conditions for some distribution to belong to the class $\mathscr{S}$ may be found, for example, in [6, 14, 18]. The class $\mathscr{S}$ includes, in particular, the following distributions on $[0,\infty)$: (i) any distribution G whose tail $\overline{G}(x)$ is regularly varying at infinity with index $\alpha < 0$, that is, for any fixed $t > 0$, $\overline{G}(xt) \sim t^\alpha \overline{G}(x)$ as $x \to \infty$; (ii) the lognormal distribution with the density $e^{-(\ln x - \ln \alpha)^2/2\sigma^2}/x\sqrt{2\pi\sigma^2}$ with $\alpha > 0$; (iii) the Weibull distribution with the tail $\overline{G}(x) = e^{-x^\alpha}$ with $\alpha \in (0,1)$.

It is known (see [19] and [10]) that, if $\mathbf{E}\xi = -a$ is finite negative number and the integrated tail distribution F^I,

$$\overline{F^I}(x) = \min\Big(1, \int_0^\infty \overline{F}(x+u)\,du\Big), \quad x > 0, \tag{3}$$

is subexponential, then the distribution tail of the maximum of sums is equivalent, up to a constant, to the integrated tail of the distribution of one summand, that is,

$$\mathbf{P}\{M > x\} \sim \overline{F^I}(x)/a \quad \text{as } x \to \infty. \tag{4}$$

The converse is also true (see [16]): if the asymptotic (4) holds, then the integrated tail distribution F^I is subexponential.

In the present paper, we consider mainly the case where the ξ's have infinite mean. In this case, we should assume $\mathbf{E}\xi^- = \infty$, otherwise $M = \infty$. Without further assumptions, we can provide lower and upper bounds only.

Theorem 1. Suppose $\mathbf{E}\xi^- = \infty$ and the condition (1) holds. Let the distribution F be long-tailed and the distribution G_1 with the tail

$$\overline{G}_1(x) = \min\left(1, \int_0^\infty \overline{F}(x+t)\, d\frac{t}{m(t)}\right) \tag{5}$$

be subexponential. Then the following estimates hold:

$$1 \le \liminf_{x\to\infty} \frac{\mathbf{P}\{M > x\}}{\overline{G}_1(x)} \le \limsup_{x\to\infty} \frac{\mathbf{P}\{M > x\}}{\overline{G}_1(x)} \le 2.$$

In the case where the function $m(x)$ is regularly varying, we get the following sharp asymptotics (the symbol Γ stands for the *Gamma function*):

Theorem 2. Suppose $\mathbf{E}\xi^- = \infty$ and the condition (1) holds. Let $m(x)$ be regularly varying at infinity with index $1-\alpha \in [0,1]$. If the distribution F is long-tailed and the distribution G_1 with the tail (5) is subexponential, then

$$\mathbf{P}\{M > x\} \sim \frac{\overline{G}_1(x)}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \quad \text{as } x \to \infty. \tag{6}$$

If $\alpha \in (0,1]$, then the assumption of the subexponentiality of G_1 can be replaced by that of the subexponentiality of the distribution G_2 with tail

$$\overline{G}_2(x) = \min\left(1, \int_1^\infty \frac{\overline{F}(x+t)}{m(t)}\, dt\right), \tag{7}$$

and then

$$\mathbf{P}\{M > x\} \sim \frac{\overline{G}_2(x)}{\Gamma(\alpha)\Gamma(2-\alpha)} \quad \text{as } x \to \infty. \tag{8}$$

The proofs of Theorems 1 and 2 are given in Section 4. Theorem 2 answers some questions on the behaviour of the maximums of sums of independent random variables raised by E. B. Dynkin in [7, § 7]. Some related results for Lévy processes can be found in [15].

Both the tails (5) and (7) are lighter than the integrated tail $\overline{F^I}$ (if the latter exists).

If both the tail $\overline{F}(t)$ and the function $m(t)$ are regularly varying at infinity, we can specify the assertion of Theorem 2 in the following way (the corresponding calculations are carried out in Section 4):

Corollary 1. Suppose $\mathbf{E}\xi^- = \infty$ and the condition (1) holds. Let $\overline{F}(t) = t^{-\beta}L^*(t)$ and $m(t) = t^{1-\alpha}L_*(t)$, where $L^*(t)$ and $L_*(t)$ are functions that are slowly varying at infinity, $0 < \alpha \le 1$, $\alpha \le \beta$. If $\alpha < \beta$, then

$$\mathbf{P}\{M > x\} \sim \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(2-\alpha)} \frac{x\overline{F}(x)}{m(x)}. \tag{9}$$

If $\alpha = \beta$, then

$$\mathbf{P}\{M > x\} \sim \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_x^\infty \frac{\overline{F}(t)}{m(t)}\,dt \equiv \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_x^\infty \frac{L^*(t)}{tL_*(t)}\,dt. \tag{10}$$

Remark 1. Let $\alpha \in [0,1)$ and $L(x)$ be a slowly varying at infinity function. Then $m(x) \sim x^{1-\alpha}L(x)$ as $x \to \infty$ if and only if $F(-x) \sim (1-\alpha)x^{-\alpha}L(x)$ (see [13, Chapter XIII, Section 5]).

An asymptotic equivalence like (9) for $\alpha \in (0,1)$, $\alpha < \beta$ is established in [5, Theorem 4.1] by other methods and under some additional technical assumptions. With regard to (10), note that, for any fixed $A > 0$,

$$\int_x^\infty \frac{L^*(t)}{tL_*(t)}\,dt \sim \int_{Ax}^\infty \frac{L^*(t)}{tL_*(t)}\,dt \quad \text{as } x \to \infty,$$

since, by the Uniform Convergence Theorem for regularly varying functions (see Theorem 1.5.2 in [4]) and by Karamata's Theorem (see Proposition 1.5.9b in [4])

$$\int_x^{Ax} \frac{L^*(t)}{tL_*(t)}\,dt \sim \frac{L^*(x)}{L_*(x)} \ln A = o\Big(\int_x^\infty \frac{L^*(t)}{tL_*(t)}\,dt\Big). \tag{11}$$

Sufficient conditions for the subexponentiality of the distributions (5) and (7) are given in Section 5. In particular, G_1 and G_2 are subexponential distributions if F is either a Pareto, Log-normal or Weibull distribution. However, in general, the subexponentiality of F only does not imply that of G_1 and G_2 (see Section 6).

The paper is organized as follows. In Sections 2 and 3, we prove some auxiliary results concerning the first descending and ascending ladder heights of a random walk. In Section 4, we give the proofs of the theorems concerning the asymptotics for $\mathbf{P}\{M > x\}$. Sufficient conditions for the subexponentiality of (5) and (7) may be found in Section 5. Finally, Section 6 is devoted to examples.

2. Asymptotics and bounds for the first descending ladder height in the infinite mean case

Let $\eta_* = \min\{n \geq 1 : S_n \leq 0\}$ be the first descending ladder epoch (we put $\min\varnothing = \infty$) and $\chi_* = -S_{\eta_*}$ be the corresponding descending ladder height. Since M is finite, η_* and χ_* are proper random variables. Moreover (see, e.g., Theorem 2.3(c) in [1, Chapter VII]), $\mathbf{E}\eta_* < \infty$ and

$$p \equiv \mathbf{P}\{M = 0\} = 1/\mathbf{E}\eta_*. \tag{12}$$

For the stopping time η_*, we have Wald's identity $\mathbf{E}\chi_* = -\mathbf{E}\eta_*\mathbf{E}\xi$, provided the mean value of ξ is finite and negative (see Theorem 2(ii) in [13, Chapter XII, Section 2]). In our analysis of the infinite-mean case, the key role will be played by the following analogue of this identity:

Lemma 1. Suppose $\mathbf{E}\xi^- = \infty$ and the condition (1) holds. Then

$$\frac{\mathbf{E}\min\{\chi_*, x\}}{m(x)} \to \mathbf{E}\eta_* \quad \text{as } x \to \infty. \tag{13}$$

In addition, for any $x \geq 0$,

$$\mathbf{E}\min\{\chi_*, x\} \leq m(x)\mathbf{E}\eta_*. \tag{14}$$

Proof. Define the taboo renewal measure on $\mathbf{R}$

$$H^*(B) = \mathbf{I}\{0 \in B\} + \sum_{n=1}^{\infty} \mathbf{P}\{S_1 > 0,\ \ldots,\ S_n > 0,\ S_n \in B\}.$$

This measure is finite since $H^*((-\infty, 0)) = 0$ and

$$\begin{aligned} H^*([0,\infty)) &= 1 + \sum_{n=1}^{\infty} \mathbf{P}\{S_1 > 0,\ \ldots,\ S_n > 0\} \\ &= 1 + \sum_{n=1}^{\infty} \mathbf{P}\{\eta_* > n\} = \mathbf{E}\eta_* < \infty. \end{aligned} \tag{15}$$

By the total probability formula, for $u \leq 0$,

$$\mathbf{P}\{-\chi_* \leq u\} = \int_0^\infty F(u-t)H^*(dt).$$

Therefore,

$$\begin{aligned}\frac{\mathbf{E}\min\{\chi_*,\,x\}}{m(x)} &= \frac{1}{m(x)}\int_0^x \mathbf{P}\{\chi_*\ge u\}du\\ &= \frac{1}{m(x)}\int_0^x\int_0^\infty F(-u-t)\,H^*(dt)\,du\\ &= \int_0^\infty \frac{m(x+t)-m(t)}{m(x)}\,H^*(dt). \qquad (16)\end{aligned}$$

For any fixed $z\ge 0$, the function $\min\{z,x\}$ is concave in $x>0$. Hence, the function $m(x)=\mathbf{E}\min\{\xi^-,x\}$ is concave as well. In particular, the function $m(x)$ is long-tailed. Taking into account also that $m(x)\to\infty$ as $x\to\infty$ (since $\mathbf{E}\xi^-=\infty$), we deduce the convergence, for any fixed $t\ge 0$,

$$\frac{m(x+t)-m(t)}{m(x)}\to 1 \quad \text{as } x\to\infty.$$

By $m(0)=0$ and by the concavity of $m(x)$,

$$\frac{m(x+t)-m(t)}{m(x)}=\frac{m(x+t)-m(t)}{m(x)-m(0)}\le 1. \qquad (17)$$

Applying now the dominated convergence theorem to the finite measure H^*, we obtain the following convergence of the integrals, as $x\to\infty$:

$$\int_0^\infty \frac{m(x+t)-m(t)}{m(x)}\,H^*(dt)\to\int_0^\infty H^*(dt)=H^*([0,\infty))=\mathbf{E}\eta_*,$$

by (15). Together with (16), this implies the convergence (13). The inequality (14) follows from (17) and (16). The proof is complete.

Let $\chi_{*1},\ \chi_{*2},\ \ldots$ be independent copies of χ_*. Define a renewal measure on $\mathbf{R}^+$

$$H_*(B)\equiv\mathbf{I}\{0\in B\}+\sum_{n=1}^\infty\mathbf{P}\{\chi_{*1}+\cdots+\chi_{*n}\in B\}.$$

If $\mathbf{E}\xi$ is finite and negative, then $H_*([0,x])\sim x\mathbf{E}\chi_*$ as $x\to\infty$, by the Key Renewal Theorem. When $\mathbf{E}\xi$ is infinite, we know only lower and upper estimates in general:

Lemma 2 (see [12, Lemma 1] or [4, Section 8.6.3])**.** Without any assumptions, for every $x\ge 0$,

$$\frac{x}{\mathbf{E}\min\{\chi_*,x\}}\le H_*([0,x])\le\frac{2x}{\mathbf{E}\min\{\chi_*,x\}}.$$

However, in the regularly varying case, the asymptotic behaviour of $H_*([0,x])$ is known:

Lemma 3 (see [11, Theorem 5])**.** If the function $\mathbf{E}\min\{\chi_*,x\}$ is regularly varying at infinity with index $1-\alpha$, $\alpha\in[0,1]$, then $H_*([0,x])$ is regularly varying at infinity with index α and

$$H_*([0,x]) \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)}\cdot\frac{x}{\mathbf{E}\min\{\chi_*,x\}} \quad \text{as } x\to\infty.$$

Using Lemma 1 and the equality (12), we obtain from Lemmas 2 and 3 the following corollaries.

Corollary 2. Suppose $\mathbf{E}\xi^-=\infty$ and the condition (1) holds. Then

$$p \le \liminf_{x\to\infty}\frac{H_*([0,x])m(x)}{x} \le \limsup_{x\to\infty}\frac{H_*([0,x])m(x)}{x} \le 2p.$$

Corollary 3. Suppose $\mathbf{E}\xi^-=\infty$ and the condition (1) holds. If $m(x)$ is regularly varying at infinity with index $1-\alpha$, $\alpha\in[0,1]$, then, as $x\to\infty$,

$$H_*([0,x]) \sim \frac{p}{\Gamma(1+\alpha)\Gamma(2-\alpha)}\cdot\frac{x}{m(x)}.$$

3. Asymptotics and bounds for the first ascending ladder height in the infinite mean case

Let $\eta^*=\min\{n\ge 1: S_n>0\}$ be the first ascending ladder epoch and $\chi^*=S_{\eta^*}$ the corresponding first ascending ladder height. Since M is finite a.s., η^* and χ^* are defective random variables, i.e. $\mathbf{P}\{\eta^*<\infty\}=1-p$ by (12).

The starting point in our analysis of the distribution of χ^* is the following representation (see [13, Chapter XII, Section 3]):

$$\mathbf{P}\{\chi^*>x\} = \int_0^\infty \overline{F}(x+t)\,H_*(dt). \tag{18}$$

Lemma 4. Suppose $\mathbf{E}\xi^-=\infty$ and the condition (1) holds. If the distribution F is long-tailed, then, for any fixed $T\ge 0$,

$$\mathbf{P}\{\chi^*>x\} \sim \int_{x+T}^\infty H_*([0,t-x])\,F(dt) \quad \text{as } x\to\infty.$$

Proof. Since F is long-tailed and $H_*([0,\infty))=\infty$,

$$\overline{F}(x) = o\biggl(\int_0^\infty \overline{F}(x+t)\,H_*(dt)\biggr) \quad \text{as } x\to\infty. \tag{19}$$

Integration of (18) by parts gives

$$\mathbf{P}\{\chi^* > x\} = \overline{F}(x+t)H_*([0,t])\Big|_0^\infty + \int_x^\infty H_*([0,t-x])\,d_tF(t). \tag{20}$$

Using the upper bound of Corollary 2, we obtain, for sufficiently large t,

$$\overline{F}(x+t)H_*([0,t]) \le \overline{F}(t)H_*([0,t]) \le 3p\overline{F}(t)\frac{t}{m(t)} = 3p\int_t^\infty \frac{t}{m(t)}\,F(ds).$$

Since the function $\frac{x}{m(x)}$ is increasing (see (2)),

$$\overline{F}(x+t)H_*([0,t]) \le 3p\int_t^\infty \frac{s}{m(s)}\,F(ds) \to 0 \quad \text{as } t\to\infty,$$

due to condition (1). Substituting this into (20), we arrive at the equality (recall that $H_*(\{0\})=1$)

$$\mathbf{P}\{\chi^* > x\} = -\overline{F}(x) + \int_x^\infty H_*([0,t-x])\,F(dt).$$

Applying now the relation (19), we deduce the equivalence of the lemma.

In the same way we obtain the following

Lemma 5. Suppose $\mathbf{E}\xi^- = \infty$ and the condition (1) holds. If the distribution F is long-tailed, then, for any fixed $T \ge 0$,

$$\int_0^\infty \overline{F}(x+t)\,d\frac{t}{m(t)} \sim \int_{x+T}^\infty \frac{t-x}{m(t-x)}\,F(dt) \quad \text{as } x\to\infty.$$

Lemma 6. Suppose $\mathbf{E}\xi^- = \infty$ and the condition (1) holds. If the distribution F is long-tailed, then

$$p \le \liminf_{x\to\infty} \frac{\mathbf{P}\{\chi^*>x\}}{\int_0^\infty \overline{F}(x+t)\,d\frac{t}{m(t)}} \le \limsup_{x\to\infty} \frac{\mathbf{P}\{\chi^*>x\}}{\int_0^\infty \overline{F}(x+t)\,d\frac{t}{m(t)}} \le 2p.$$

Proof. Fix $\varepsilon > 0$. It follows from Corollary 2 that there exists $T > 0$ such that, for $t > T$,

$$(p-\varepsilon)\frac{t}{m(t)} \le H_*([0,t]) \le (2p+\varepsilon)\frac{t}{m(t)}.$$

Applying Lemma 4, we obtain, for x sufficiently large,

$$(p-2\varepsilon)\int_{x+T}^\infty \frac{t-x}{m(t-x)}\,F(dt) \le \mathbf{P}\{\chi^*>x\} \le (2p+2\varepsilon)\int_{x+T}^\infty \frac{t-x}{m(t-x)}\,F(dt).$$

The asymptotic equivalence in Lemma 5 completes the proof, since $\varepsilon > 0$ was choosen arbitrary.

Using Corollary 3 instead of Corollary 2, we may deduce the following

Lemma 7. Suppose $\mathbf{E}\xi^- = \infty$ and the condition (1) holds. Let the function $m(x)$ be regularly varying at infinity with index $1-\alpha$, $\alpha \in [0,1]$. If the distribution F is long-tailed, then

$$\mathbf{P}\{\chi^* > x\} \sim \frac{p}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \int_0^\infty \overline{F}(x+t)\, d\frac{t}{m(t)} \qquad \text{as } x \to \infty.$$

Proof. It follows from Corollary 3 that there exists $T > 0$ such that, for $t > T$,

$$\frac{p-\varepsilon}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \frac{t}{m(t)} \le H_*([0,t]) \le \frac{p+\varepsilon}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \frac{t}{m(t)}.$$

By Lemma 4, for x sufficiently large,

$$\frac{p-2\varepsilon}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \int_{x+T}^\infty \frac{t-x}{m(t-x)} F(dt) \le \mathbf{P}\{\chi^* > x\} \le \frac{p+2\varepsilon}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \int_{x+T}^\infty \frac{t-x}{m(t-x)} F(dt).$$

Applying Lemma 5 completes the proof.

4. The asymptotics and bounds for the distribution tail of the supremum

We start with a general theorem which describes the tail behaviour of the supremum in terms of the renewal measure H_*.

Theorem 3. Suppose $\mathbf{E}\xi^- = \infty$ and the condition (1) holds. Let the distribution F be long-tailed and the distribution G with the tail

$$\overline{G}(x) = \min\left(1, \int_0^\infty \overline{F}(x+t)\, d\frac{t}{m(t)}\right)$$

be subexponential. Then, as $x \to \infty$,

$$\mathbf{P}\{M > x\} \sim \frac{1}{p} \int_0^\infty \overline{F}(x+t)\, H_*(dt).$$

Proof. Consider the distribution G_H with the tail

$$\overline{G_H}(x) = \min\left(1, \int_0^\infty \overline{F}(x+t)\, H_*(dt)\right).$$

This distribution is long-tailed because F is long-tailed. In addition, by Lemma 6, the tail of G_H is sandwiched asymptotically between the subexponential tails

$p\overline{G}$ and $2p\overline{G}$. Therefore, by the weak equivalence property (see Theorem 2.1 in [14] or Lemma 1 in [3]) the distribution G_H is subexponential as well.

Let us define non-defective random variable $\widetilde{\chi}$ with distribution on $(0,\infty)$

$$\mathbf{P}\{\widetilde{\chi}\in B\} = \frac{\mathbf{P}\{\chi^*\in B\}}{1-p}.$$

The distribution of $\widetilde{\chi}$ is subexponential. Let $\widetilde{\chi}_1$, $\widetilde{\chi}_2$, ... be independent copies of the random variable $\widetilde{\chi}$. Notice that there exists $i\ge 1$ such that S_i exceeds a level x if and only if one of the ladder heights exceeds this level. Hence, by the formula of total probability we have the equality:

$$\mathbf{P}\{M\in B\} = \sum_{n=1}^{\infty}(1-p)^n p\,\mathbf{P}\{\widetilde{\chi}_1+\cdots+\widetilde{\chi}_n\in B\}.$$

Since the random variable $\widetilde{\chi}$ has a subexponential distribution, we may apply the stopping time theorem (see, e.g., Lemma 1.8 [2, Chapter IX, Section 1] or Lemma 1.3.5 [8, Section 1.3.2]) and write

$$\mathbf{P}\{M>x\}\sim\mathbf{P}\{\widetilde{\chi}>x\}\sum_{n=1}^{\infty}(1-p)^n p\,n = \frac{\mathbf{P}\{\chi^*>x\}}{p}.$$

The proof is complete.

The latter result looks strange in the sense that while the conditions are expressed in terms of the reference distribution F, the resulting integral is taken with respect to the renewal measure which is a rather complicated object. In general, we are unable to write the asymptotics for the integral

$$\int_0^\infty \overline{F}(x+t)\,H_*(dt)$$

in terms of the distribution F itself, due to the lack of the information about the asymptotic behaviour of the renewal function $H_*([0,x])$ as $x\to\infty$ in the case of infinite mean. We may deduce the lower and upper bounds only: combining the asymptotics in Theorem 3 and the bounds in Lemma 6, we get the assertion of Theorem 1.

To the best of our knowledge, the case when the function $m(x)$ is regularly varying is the only one where the asymptotic behaviour of $H_*([0,x])$ is known. In this case, combining Theorem 3 and Lemma 7, we obtain the relation (6) of Theorem 2.

For $\alpha\in[0,1]$, $tF(-t)=(1-\alpha+o(1))m(t)$ as $t\to\infty$. Thus, it follows from (2) that

$$\frac{d}{dt}\frac{t}{m(t)} = \frac{\alpha+o(1)}{m(t)} \quad \text{as } t\to\infty.$$

For $\alpha \in (0,1]$, we can apply this result to deduce (8) from (6).

Finally, we prove Corollary 1. Notice that the distribution (5) is subexponential in this case by Lemma 8 from the next section. We start with the case $0 < \alpha \le 1$, $\alpha < \beta$. Fix $\varepsilon > 0$ and $A > 0$. We have

$$\int_1^{\varepsilon x} \frac{\overline{F}(x+t)}{m(t)}\,dt \le \overline{F}(x)\int_1^{\varepsilon x}\frac{1}{m(t)}\,dt \sim \frac{\overline{F}(x)}{\alpha}\frac{\varepsilon x}{m(\varepsilon x)} \quad \text{as } x\to\infty \tag{21}$$

and

$$\int_{Ax}^{\infty} \frac{\overline{F}(x+t)}{m(t)}\,dt \le \int_{Ax}^{\infty}\frac{\overline{F}(t)}{m(t)}\,dt \sim \frac{1}{\beta-\alpha}\frac{Ax\overline{F}(Ax)}{m(Ax)} \quad \text{as } x\to\infty. \tag{22}$$

Next,

$$\begin{aligned}\int_{\varepsilon x}^{Ax} \overline{F}(x+t)\frac{1}{m(t)}\,dt &= \frac{\overline{F}(x)}{m(x)}\int_{\varepsilon x}^{Ax}\frac{\overline{F}(x+t)}{\overline{F}(x)}\frac{m(x)}{m(t)}\,dt\\ &= \frac{x\overline{F}(x)}{m(x)}\int_{\varepsilon}^{A}\frac{\overline{F}(x(1+s))}{\overline{F}(x)}\frac{m(x)}{m(xs)}\,ds\\ &\sim \frac{x\overline{F}(x)}{m(x)}\int_{\varepsilon}^{A}\frac{(1+s)^{-\beta}}{s^{1-\alpha}}\,ds \quad \text{as } x\to\infty,\end{aligned} \tag{23}$$

since, by the Uniform Convergence Theorem for regularly varying functions (see Theorem 1.5.2 in [4])

$$\frac{\overline{F}(x(1+s))}{\overline{F}(x)}\frac{m(x)}{m(xs)} \to \frac{(1+s)^{-\beta}}{s^{1-\alpha}}$$

as $x\to\infty$ uniformly in $s\in[\varepsilon,A]$. Letting $\varepsilon\to 0$ and $A\to\infty$, we obtain from (21), (22) and (23) that

$$\mathbf{P}\{M>x\} \sim \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\frac{x\overline{F}(x)}{m(x)}\int_0^{\infty}(1+s)^{-\beta}s^{\alpha-1}\,ds = \frac{B(\beta-\alpha,\alpha)}{\Gamma(\alpha)\Gamma(2-\alpha)}\frac{x\overline{F}(x)}{m(x)},$$

which implies (9); here B is the *Beta function.*

We now consider the case $0<\alpha\le 1$, $\alpha=\beta$. Fix $A>0$. Now we have

$$\begin{aligned}\int_1^{Ax}\frac{\overline{F}(x+t)}{m(t)}\,dt &\le \overline{F}(x)\int_1^{Ax}\frac{dt}{m(t)} \sim \frac{\overline{F}(x)}{\alpha}\frac{Ax}{m(Ax)}\\ &\sim \frac{A^{\alpha}}{\alpha}\frac{L^*(x)}{L_*(x)} = o\Big(\int_x^{\infty}\frac{L^*(t)}{tL_*(t)}\,dt\Big),\end{aligned} \tag{24}$$

since $\alpha=\beta$ and using (11). Further, for any small $\delta>0$ there exists A sufficiently large such that $\overline{F}(x+t)\ge(1-\delta)\overline{F}(t)$ for any $t\ge Ax$. Then

$$\int_{Ax}^{\infty}\frac{\overline{F}(x+t)}{m(t)}\,dt \ge (1-\delta)\int_{Ax}^{\infty}\frac{\overline{F}(t)}{m(t)}\,dt = (1-\delta)\int_{Ax}^{\infty}\frac{L^*(t)}{tL_*(t)}\,dt. \tag{25}$$

On the other hand,

$$\int_{Ax}^{\infty} \frac{\overline{F}(x+t)}{m(t)}\,dt \le \int_{Ax}^{\infty} \frac{\overline{F}(t)}{m(t)}\,dt = \int_{Ax}^{\infty} \frac{L^*(t)}{tL_*(t)}\,dt. \tag{26}$$

The relations (24), (25), (26) and (11) imply (10). Corollary 1 is proved.

5. Sufficient conditions for the integrated weighted tail distribution to be subexponential

In this Section, we present sufficient conditions for the subexponentiality of the distributions (5) and (7). We consider an even more general problem: Let F be a distribution on $\mathbf{R}^+$ and H a non-negative measure on $\mathbf{R}^+$ such that

$$\int_0^{\infty} \overline{F}(t)\,H(dt) \quad \text{is finite.} \tag{27}$$

In this case we can define the distribution G_H with the tail

$$\overline{G}_H(x) \equiv \min\left(1, \int_0^{\infty} \overline{F}(x+t)\,H(dt)\right), \quad x \ge 0. \tag{28}$$

We can formulate the following question: what type of conditions on F imply the subexponentiality of G_H?

First, recall that if F is long-tailed, then G_H is long-tailed as well.

Definition 3. The distribution F on $\mathbf{R}^+$ is called dominated varying ($F \in \mathscr{D}$) iff, for some $c > 0$, $\overline{F}(2x) \ge c\overline{F}(x)$ for any x.

It is known that $(\mathscr{L} \cap \mathscr{D}) \subset \mathscr{S}$. Also, it is known that if $F \in \mathscr{D}$, then $F^I \in \mathscr{L} \cap \mathscr{D}$, but the converse is not true in general (see [14, Section 4]).

Lemma 8. If $F \in \mathscr{D} \cap \mathscr{L}$, then $G_H \in \mathscr{D} \cap \mathscr{L}$ and, therefore, $G_H \in \mathscr{S}$.

Proof. This result follows from the inequalities:

$$\int_0^{\infty} \overline{F}(2x+t)\,H(dt) \ge c\int_0^{\infty} \overline{F}(x+t/2)\,H(dt) \ge c\int_0^{\infty} \overline{F}(x+t)\,H(dt).$$

Definition 4. The distribution F on $\mathbf{R}^+$ with finite mean m belongs to the class $\mathscr{S}^*$ if

$$\int_0^{x} \overline{F}(x-y)\overline{F}(y)dy \sim 2m\overline{F}(x) \quad \text{as } x \to \infty.$$

It is known (see [14]) that

$$F \in \mathscr{S}^* \quad \text{implies} \quad F \in \mathscr{S} \quad \text{and} \quad F^I \in \mathscr{S}. \tag{29}$$

It turns out that the following more general conclusion holds. For any $b > 0$, define the class $\mathscr{H}_b$ of all non-negative measures H on $\mathbf{R}^+$ such that

$$\sup_t H((t, t+1]) \leq b.$$

Lemma 9. Let $F \in \mathscr{S}^*$ and $H \in \mathscr{H}_b$, $b \in (0, \infty)$. Then $G_H \in \mathscr{S}$. Moreover,

$$\overline{G_H * G_H}(x) \sim 2\overline{G}_H(x)$$

as $x \to \infty$ uniformly in $H \in \mathscr{H}_b$.

Remark 2. Here are four examples of such measures H: (i) if $H(B) = \mathbf{I}\{0 \in B\}$, then $G_H = F$; (ii) if $H(dt) = dt$ is Lebesgue measure on $\mathbf{R}^+$, then $G_H = F^I$; (iii) if H is the renewal measure H_*, then G_H is the distribution of the first ascending ladder height χ^*; (iv) if $H([0, x]) = x/m(x)$, then G_H is G_1 from (5).

Remark 3. Lemma 9 also implies that any $\mathscr{S}^*$-distribution is strongly subexponential in the sense of [17] (see definition 3 of that paper).

Remark 4. It is natural to consider the following two questions:

(i) may the assumption $F \in \mathscr{S}^*$ of Lemma 9 be weakened to $F \in \mathscr{S}$? In the case of Lebesgue measure H, i.e. when $G_H = F^I$, this question is raised in [8, Section 1.4.2].

(ii) is the converse of (29) also true?

In the next Section, we show (by examples) that the answers to both these questions are negative.

Proof of Lemma 9. Since G_H is long-tailed uniformly in $H \in \mathscr{H}_b$, it is sufficient to show that

$$\lim_{A\to\infty} \limsup_{x\to\infty} \sup_{H\in\mathscr{H}_b} \frac{1}{\overline{G}_H(x)} \int_A^{x-A} \overline{G}_H(x-y) G_H(dy) = 0, \tag{30}$$

see, e.g., Proposition 2 in [3].

The mean value of F is finite. Thus, $\overline{F}(t)H((0,t]) = o(1/t)O(t) \to 0$ as $t \to \infty$ and integration by parts yields, for x large enough,

$$\overline{G}_H(x) = \int_x^\infty H((0, t-x])F(dt).$$

Hence,

$$G_H((x, x+1]) = \int_x^\infty H((t-x-1, t-x])\, F(dt) \leq b\overline{F}(x).$$

In addition, G_H is long-tailed. Therefore, (30) holds if and only if

$$\lim_{A\to\infty} \limsup_{x\to\infty} \sup_{H\in\mathscr{H}_b} \frac{1}{\overline{G}_H(x)} \int_A^{x-A} \overline{G}_H(x-y) \overline{F}(y)\, dy = 0. \tag{31}$$

Fix $\varepsilon > 0$. Since $F \in \mathscr{S}^*$, there exist x_0 and A such that, for all $x \ge x_0$,

$$\int_A^{x-A} \overline{F}(x-u)\overline{F}(u)\,du \le \varepsilon \overline{F}(x).$$

Then, for $x \ge x_0$,

$$\begin{aligned}\int_A^{x-A} \overline{G}_H(x-y)\overline{F}(y)\,dy &= \int_A^{x-A}\left(\int_0^\infty \overline{F}(x+t-y)\,H(dt)\right)\overline{F}(y)dy \\ &\le \int_0^\infty\left(\int_A^{x+t-A}\overline{F}(x+t-y)\overline{F}(y)dy\right)H(dt) \\ &\le \varepsilon\int_0^\infty \overline{F}(x+t)H(dt) = \varepsilon\overline{G}_H(x).\end{aligned}$$

Letting ε to 0, we get (31). The proof is complete.

6. Examples

In this Section, we give an example of $F \in \mathscr{S}$ with finite mean such that $F^I \notin \mathscr{S}$. In fact, we provide a more general example: for any fixed $\alpha \in [0,1)$, we construct a subexponential distribution F with finite mean such that the distribution G_α with the tail

$$\overline{G_\alpha}(x) = \min\left(1, \int_1^\infty \frac{\overline{F}(x+y)}{y^\alpha}\,dy\right) \tag{32}$$

is not subexponential. In particular, when $\alpha = 0$, F^I does not belong to $\mathscr{S}$.

In our second example, we show that two conditions $F \in \mathscr{S}$ and $F^I \in \mathscr{S}$ taken together do not imply that $F \in \mathscr{S}^*$.

Both examples are based on the following construction.

Define two increasing sequences of positive numbers, namely $\{t_n\}$ and $\{R_n\}$, such that, as $n \to \infty$,

$$t_n = o(t_{n+1}), \tag{33}$$

$$R_{n+1} - R_n \to \infty. \tag{34}$$

Define the *hazard function* $R(x) \equiv -\ln \overline{F}(x)$ as

$$R(x) = R_n + r_n(x-t_n) \quad \text{for} \quad t_n \le x \le t_{n+1},$$

where

$$r_n = \frac{R_{n+1}-R_n}{t_{n+1}-t_n} \sim \frac{R_{n+1}-R_n}{t_{n+1}}. \tag{35}$$

by (33) and (34). In other words, the *hazard rate* $r(x) \equiv R'(x)$ is defined as $r(x) = r_n$ for $x \in (t_n, t_{n+1}]$, where r_n is given by (35).

Note that

$$J_n \equiv \int_{t_n}^{t_{n+1}} \overline{F}(u)du = \int_{t_n}^{t_{n+1}} e^{-R(u)}du = \frac{e^{-R_n} - e^{-R_{n+1}}}{r_n} < \frac{e^{-R_n}}{r_n},$$

and that the mean value of F is finite provided

$$\sum_n \frac{e^{-R_n}}{r_n} < \infty. \tag{36}$$

We assume that (36) holds. Assume also that

$$r_{n+1} = o(r_n) \text{ and } r_n t_n \to 0 \text{ as } n \to \infty. \tag{37}$$

It follows from (37) that $r_k t_n \to 0$ as $n \to \infty$ uniformly in $k \geq n$. However,

$$r_n t_{n+1} \sim R_{n+1} - R_n \to \infty \tag{38}$$

from (35) and (34). It follows from (37) that $r(x)$ decreases eventually to 0, and we can apply the following results:

Proposition 1 (see Corollary 3.8 and Theorem 3.6 in [14]). If the hazard rate exists and is eventually decreasing to 0, then $F \in \mathscr{L}$ and

(i) $F \in \mathscr{S}$ if and only if

$$\lim_{x\to\infty} r(x) \int_0^x e^{yr(x)}\overline{F}(y)dy = 0. \tag{39}$$

(ii) $F \in \mathscr{S}^*$ if and only if F has finite mean and

$$\lim_{x\to\infty} \int_0^x e^{yr(x)}\overline{F}(y)\,dy = \int_0^\infty \overline{F}(y)\,dy. \tag{40}$$

Note that (40) is equivalent to

$$\lim_{t\to\infty} \lim_{x\to\infty} \int_t^x e^{yr(x)}\overline{F}(y)\,dy = 0. \tag{41}$$

Put

$$I_{n,k} = \int_{t_k}^{t_{k+1}} e^{yr_n}\overline{F}(y)\,dy \quad \text{and} \quad I_n = \int_1^{t_{n+1}} e^{yr_n}\overline{F}(y)\,dy = \sum_{k=1}^n I_{n,k}.$$

In our case, (39) holds if and only if

$$r_n I_n \to 0 \text{ as } n \to \infty. \tag{42}$$

The relation (41) fails, in particular, if

$$\liminf_{n\to\infty} I_{n,n} > 0. \tag{43}$$

From (37),

$$I_{n,k} = e^{-R_k+r_kt_k}\frac{e^{(r_n-r_k)t_k}-e^{(r_n-r_k)t_{k+1}}}{r_k-r_n} \le \frac{e^{-R_k+r_nt_k}}{r_k-r_n} \sim \frac{e^{-R_k}}{r_k}$$

as $k\to\infty$ uniformly in $n\ge k+1$. Thus, for some $C<\infty$,

$$\sum_{k=1}^{n-1} I_{n,k} \le C\sum_{k=1}^{n-1}\frac{e^{-R_k}}{r_k} \le C\sum_{k=1}^{\infty}\frac{e^{-R_k}}{r_k} < \infty \tag{44}$$

if (36) holds. Further,

$$r_nI_{n,n} = r_n(t_{n+1}-t_n)e^{-R_n+r_nt_n} \sim (R_{n+1}-R_n)e^{-R_n} \tag{45}$$

from (35) and (37). Thus, $r_nI_{n,n}\to 0$ if

$$R_{n+1} = o(e^{R_n}). \tag{46}$$

Hence, under the conditions (33), (34), (36), (37), and (46), F is a subexponential distribution with finite mean.

We now turn to the examples.

Example 1. Fix $\alpha\in[0,1)$ and put $R_{n+1}=e^{\gamma R_n}$, where the constant $\gamma=\gamma(\alpha)\in(0,1)$ will be specified later. If we take $R_1=R_1(\gamma)$ sufficiently large, then the sequence R_n will be increasing and, moreover, $R_{n+1}/R_n\to\infty$. Put $t_{n+1}=e^{2\gamma R_n}=R_{n+1}^2$; the condition (33) is satisfied. We have

$$\overline{F}(t_n) = e^{-\sqrt{t_n}}.$$

We also have $r_n\sim R_{n+1}/t_{n+1}=e^{-\gamma R_n}$.

The condition (36) is valid since $J_n\sim e^{-R_n(1-\gamma)}$. The condition (37) holds since $r_{n+1}/r_n\sim e^{-\gamma(R_{n+1}-R_n)}$ and $r_nt_n\sim e^{-\gamma R_n+2\gamma R_{n-1}}=e^{-\gamma R_n+o(R_n)}$. Finally, the condition (46) follows since $R_{n+1}e^{-R_n}=e^{-(1-\gamma)R_n}$. Hence, F has a finite mean and is subexponential.

Take now the distribution G_α defined in (32) and estimate its density. For $x\in(t_n,t_{n+1}-1]$,

$$\begin{aligned} G'_\alpha(x) &= -\int_1^\infty \frac{\overline{F}'(x+y)}{y^\alpha}\,dy = \int_1^\infty \frac{r(x+y)\overline{F}(x+y)}{y^\alpha}\,dy \\ &\ge \int_1^{t_{n+1}-x}\frac{r(x+y)\overline{F}(x+y)}{y^\alpha}\,dy = r_nV_n(x), \end{aligned} \tag{47}$$

where

$$V_n(x) = \int_1^{t_{n+1}-x} \overline{F}(x+y)y^{-\alpha}dy.$$

We also have that, for any $x < t_{n+1} - 1$,

$$\overline{G}_\alpha(x) \geq V_n(x). \tag{48}$$

For any $x \in (t_n, t_{n+1} - 1]$,

$$\overline{G_\alpha}(x) = \left(\int_{x+1}^{t_{n+1}} + \sum_{k=n+1}^{\infty} \int_{t_k}^{t_{k+1}} \right) \overline{F}(y)(y-x)^{-\alpha}dy \equiv V_n(x) + \sum_{k=n+1}^{\infty} W_k(x).$$

For $x \in (t_n, t_{n+1}]$ and $k \geq n+1$, by (37) and (38),

$$\begin{aligned} W_k(x) &= e^{-R_k + r_k t_k} \int_{t_k}^{t_{k+1}} e^{-r_k y}(y-x)^{-\alpha}dy \\ &\sim e^{-R_k} \int_{t_k}^{t_{k+1}} e^{-r_k y}(y-x)^{-\alpha}dy = \frac{e^{-R_k}}{r_k^{1-\alpha}} \int_{r_k t_k}^{r_k t_{k+1}} e^{-y}(y - r_k x)^{-\alpha}dy \\ &\sim \frac{e^{-R_k}}{r_k^{1-\alpha}} \int_0^{\infty} e^{-y}y^{-\alpha}dy = \frac{e^{-R_k}}{r_k^{1-\alpha}}\Gamma(1-\alpha) \quad \text{as } k \to \infty. \end{aligned}$$

Similarly, for $x \in (t_n, t_{n+1}/2]$,

$$\begin{aligned} V_n(x) &= e^{-R_n + r_n t_n - r_n x} \int_1^{t_{n+1}-x} e^{-r_n y}y^{-\alpha}dy \\ &\sim \frac{e^{-R_n - r_n x}}{r_n^{1-\alpha}} \int_{r_n}^{r_n(t_{n+1}-x)} e^{-z}z^{-\alpha}dz \sim \frac{e^{-R_n - r_n x}}{r_n^{1-\alpha}}\Gamma(1-\alpha) \quad \text{as } n \to \infty. \end{aligned}$$

Since $\gamma < 1$,

$$\frac{W_{k+1}(x)}{W_k(x)} \sim \left(\frac{r_k}{r_{k+1}}\right)^{1-\alpha} e^{-R_{k+1}+R_k} \sim e^{[\gamma(1-\alpha)-1](R_{k+1}-R_k)} \to 0.$$

Take any integer $l \geq 2$ such that $\gamma(1-\alpha) < (l-1)/l$. Then, as $n \to \infty$,

$$\frac{W_{n+1}(x)}{V_n(t_{n+1}/l)} \sim \left(\frac{r_n}{r_{n+1}}\right)^{1-\alpha} e^{-R_{n+1}+R_n+r_n t_{n+1}/l} = e^{[\gamma(1-\alpha)-(l-1)/l]R_{n+1}+o(R_{n+1})} \to 0.$$

Therefore,

$$\overline{G_\alpha}(t_{n+1}/l) \sim V_n(t_{n+1}/l) \sim \frac{e^{-R_n - r_n t_{n+1}/l}}{r_n^{1-\alpha}}\Gamma(1-\alpha) \quad \text{as } n \to \infty.$$

On the other hand, by (47), for n sufficiently large,

$$\overline{G_\alpha * G_\alpha}(t_{n+1}/l) \geq \int_{t_n}^{t_{n+1}/2l} \overline{G}_\alpha(t_{n+1}/l - y)\, G_\alpha(dy)$$
$$\geq r_n \int_{t_n}^{t_{n+1}/2l} \overline{G}_\alpha(t_{n+1}/l - y) V_n(y)\, dy.$$

Applying now (48), we get

$$\overline{G_\alpha * G_\alpha}(t_{n+1}/l) \geq r_n \int_{t_n}^{t_{n+1}/2l} V_n(t_{n+1}/l - y) V_n(y)\, dy$$
$$\sim \frac{\Gamma^2(1-\alpha)}{r_n^{1-2\alpha}} \int_{t_n}^{t_{n+1}/2l} e^{-R_n - r_n(t_{n+1}/l - y) - R_n - r_n y}\, dy$$
$$\sim \frac{\Gamma^2(1-\alpha)}{r_n^{1-2\alpha}} \frac{t_{n+1}}{2l} e^{-2R_n - r_n t_{n+1}/l}.$$

Then the ratio

$$\frac{\overline{G_\alpha * G_\alpha}(t_{n+1}/l)}{\overline{G_\alpha}(t_{n+1}/l)}$$

is asymptotically not less than

$$\frac{\Gamma(1-\alpha)}{2l} r_n^\alpha t_{n+1} e^{-R_n} \sim \frac{\Gamma(1-\alpha)}{2l} e^{R_n(-\gamma\alpha + 2\gamma - 1)} \to \infty$$

as $n \to \infty$ provided $\gamma(2-\alpha) > 1$. Thus, for any $\gamma \in (1/(2-\alpha), 1)$, $F \in \mathscr{S}$ and has finite mean, but $G_\alpha \notin \mathscr{S}$.

Example 2. For $\gamma > 2$, take $R_n = n^\gamma$ and $t_{n+1} = e^{R_n} = e^{n^\gamma}$. Then

$$\overline{F}(t_n) = t_n^{-\left(\frac{n}{n-1}\right)^\gamma}.$$

Conditions (33), (34), and (37) are satisfied, $r_n \sim \gamma n^{\gamma-1}/t_{n+1}$, and (36) holds. Further, (46) holds too. Hence, $F \in \mathscr{S}$.

On the other hand, for $x \in (t_n, t_{n+1}]$,

$$\overline{F^I}(x) \leq \sum_{k=n}^{\infty} J_k \quad \text{and} \quad \overline{F^I}(2x) \geq \sum_{k=n+2}^{\infty} J_k.$$

In addition, $J_k \sim 1/\gamma k^{\gamma-1}$ as $k \to \infty$. Thus, $\overline{F^I}(2x) \sim \overline{F^I}(x)$ as $x \to \infty$ and the function $\overline{F^I}(x)$ is slowly varying at infinity. Hence, $F^I \in \mathscr{S}$.

However, from (45) and (35),

$$I_{n,n} \sim t_{n+1} e^{-R_n} = 1$$

and so, from (43), F cannot belong to $\mathscr{S}^*$.

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