ASYMPTOTICS OF RANDOMLY STOPPED SUMS IN THE PRESENCE OF HEAVY TAILS

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Abstract

We study conditions under which

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\mathbf{P}\{\xi_1 > x\} \text{ as } x \to \infty,$$

where S_τ is a sum $\xi_1 + \ldots + \xi_\tau$ of random size τ and M_τ is a maximum of partial sums $M_\tau = \max_{n \le \tau} S_n$. Here ξ_n, $n = 1, 2, \ldots$, are independent identically distributed random variables whose common distribution is assumed to be subexponential. We consider mostly the case where τ is independent of the summands; also, in a particular situation, we deal with a stopping time.

Also we consider the case where $\mathbf{E}\xi > 0$ and where the tail of τ is comparable with or heavier than that of ξ, and obtain the asymptotics

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\mathbf{P}\{\xi_1 > x\} + \mathbf{P}\{\tau > x/\mathbf{E}\xi\} \text{ as } x \to \infty.$$

This case is of a primary interest in the branching processes.

In addition, we obtain new uniform (in all x and n) upper bounds for the ratio $\mathbf{P}\{S_n > x\}/\mathbf{P}\{\xi_1 > x\}$ which substantially improve Kesten's bound in the subclass $\mathscr{S}^*$ of subexponential distributions.

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1. Introduction

Let ξ, ξ_1, ξ_2, ... be independent identically distributed random variables with a finite mean. We assume that their common distribution F is right-unbounded, that is, $\overline{F}(x) \equiv \mathbf{P}\{\xi > x\} > 0$ for all x. Moreover, we assume that F has a *heavy* (right) tail. Recall that a random variable η has a *heavy-tailed* distribution if $\mathbf{E}e^{\varepsilon\eta} = \infty$ for all $\varepsilon > 0$, and *light-tailed* otherwise.

Let $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n = 1, 2, \ldots$, and let $M_n = \max_{0 \le i \le n} S_i$ be the partial maxima. Denote by F^{*n} the distribution of S_n.

Let τ be a counting random variable with a finite mean. In this paper, we study the asymptotics for the tail probabilities $\mathbf{P}\{S_\tau > x\}$ and $\mathbf{P}\{M_\tau > x\}$ as $x \to \infty$.

It is known that, for *any* distribution F on $\mathbf{R}^+$ and for *any* counting random variable τ which is independent of the sequence $\{\xi_n\}$,

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \ge \mathbf{E}\tau,$$

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see, e.g. [35, 10]. It was proved in the series of papers [13, 9, 10] that if F is a heavy-tailed distribution on $\mathbf{R}^+$ with finite mean and if $\mathbf{P}\{c\tau > x\} = o(\overline{F}(x))$ as $x \to \infty$, for some $c > \mathbf{E}\xi$, then

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} = \mathbf{E}\tau. \tag{1}$$

This gives us the idea what asymptotic behaviour of $\mathbf{P}\{S_\tau > x\}$ should be expected, at least if the tail of τ is lighter than that of ξ. In particular, by considering the case $\tau = 2$, we conclude that if F is a heavy-tailed distribution on $\mathbf{R}^+$ and if $\mathbf{P}\{S_2 > x\} \sim c\overline{F}(x)$ as $x \to \infty$, for some c, then $c = 2$ with necessity (see [13]). By the latter observation, we restrict our attention to subexponential distributions only.

A distribution F on $\mathbf{R}^+$ with unbounded support is called *subexponential*, $F \in \mathscr{S}$, if $\overline{F*F}(x) \sim 2\overline{F}(x)$ as $x \to \infty$. A distribution F on $\mathbf{R}$ is called subexponential if its conditional distribution on $\mathbf{R}^+$ is subexponential. It is well known that any subexponential distribution is heavy-tailed and, even more, is long-tailed. A distribution F with right-unbounded support is called *long-tailed* if $\overline{F}(x+y) \sim \overline{F}(x)$ as $x \to \infty$, for any fixed y.

The key result in the theory of subexponential distributions is: if F is subexponential and if τ does not depend on the summands and is light-tailed, then

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x) \ \text{ as } x \to \infty. \tag{2}$$

A converse result also holds: if, for a distribution F on $\mathbf{R}^+$ and for an independent counting random variable $\tau \geq 2$, $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ as $x \to \infty$, then F is subexponential (see, e.g. [11]).

The intuition behind relation (2) is the *principle of one big jump*: in the case of heavy tails, for x large, the most probable way leading to the event $\{S_n > x\}$ is that one of n summands $\xi_1, \ldots, \xi_n$ is large while all others are relatively small. Asymptotically this gives the probability $n\overline{F}(x)$, and conditioning on τ yields to the multiplier $\mathbf{E}\tau$. The keystone of the proof is Kesten's bound: for any subexponential distribution F and for any $\varepsilon > 0$, there exists $K = K(F, \varepsilon)$ such that the inequality

$$\overline{F^{*n}}(x) \leq K(1+\varepsilon)^n\overline{F}(x)$$

holds for all x and n; see, e.g. [2, Section IV.4]. Clearly this estimate does not help to prove (2) if the distribution of τ is heavy-tailed. So the question of the basic importance is: If we fix a subexponential distribution F, then what are the weakest natural conditions on τ which still guarantee relation (2) to hold? Intuitively, the light-tailedness assumption seems to be very strong. The study of this problem is one of the main topics of the present paper.

In order to state our first result, we need to introduce the notion of $\mathscr{S}^*$-distribution. A distribution F on $\mathbf{R}$ with a finite mean belongs to the class $\mathscr{S}^*$ if

$$\int_0^x \overline{F}(x-y)\overline{F}(y)dy \sim 2a\overline{F}(x) \quad \text{ as } x \to \infty,$$

where $a = 2\int_0^\infty \overline{F}(y)dy$. It is known (see Klüppelberg [18]) that any distribution from the class $\mathscr{S}^*$ is subexponential. Though these two classes, $\mathscr{S}^*$ and $\mathscr{S}$, are considered as rather similar, there exist subexponential distributions which are not in $\mathscr{S}^*$, see, e.g. [8] and the discussion in Section 2 below. Classical examples of distributions from the class $\mathscr{S}^*$ are Pareto, log-normal, and Weibull with parameter $\beta \in (0, 1)$.

Theorem 1. *Assume that a counting random variable τ does not depend on $\{\xi_n\}$. Let $F \in \mathscr{S}^*$.*

(i) *If $\mathbf{E}\xi < 0$ then*

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x) \ \ \text{as } x \to \infty. \tag{3}$$

(ii) *If $\mathbf{E}\xi \geq 0$ and if there exists $c > \mathbf{E}\xi$ such that*

$$\mathbf{P}\{c\tau > x\} = o(\overline{F}(x)) \ \ \text{as } x \to \infty, \tag{4}$$

then asymptotics (3) *again hold.*

The latter theorem shows that if we restrict our attention from the class of all heavy-tailed distributions to the class $\mathscr{S}^*$, we obtain equivalence (3) which is stronger than assertion (1) for the '$\liminf$'. Definitely we should assume the subexponentiality of F in order to get (3). At the end of Section 4 we construct an example demonstrating that the stronger condition $F \in \mathscr{S}^*$ is essential for the statement to hold in the whole generality and cannot be replaced by condition $F \in \mathscr{S}$.

The proof of Theorem 1 is carried out in Section 4. Statement (i) can be found in [15]; in Section 4 we give an alternative proof of (i). Note that these two cases, negative and positive mean of ξ, are substantially different in their nature.

Condition (4) seems to be essential, since, for any $c < \mathbf{E}\xi$,

$$\begin{aligned}\mathbf{P}\{S_\tau > x\} &= \mathbf{P}\{S_\tau > x, c\tau \leq x\} + \mathbf{P}\{S_\tau > x, c\tau > x\}\\ &\geq (\mathbf{E}\tau + o(1))\overline{F}(x) + (1+o(1))\mathbf{P}\{c\tau > x\}\end{aligned}$$

as $x \to \infty$, due to the convergence $\mathbf{P}\{S_\tau > x | c\tau > x\} \to 1$, by the Law of Large Numbers. In particular, for τ with a regularly varying tail distribution, condition (4) is necessary for asymptotic relation (3) to hold. Further discussion on condition (4) can be found in Section 4.

Stam in [37, Theorem 5.1] and A. Borovkov and K. Borovkov in [3, Section 7.1] obtained asymptotics (3) under condition (4) for regularly varying F. Some results by Stam [37] have been proved again by Faÿ *et al.* in [12]. The case where F is a dominated varying distribution was studied by Ng *et al*. [30] and by Daley *et al*. [6]. A subclass of so-called semi-exponential F was considered in [3, Section 7.2]. In [15, Corollary 2], asymptotics (3) were obtained in the case $\mathbf{E}\xi \geq 0$ under the extra assumption $\mathbf{P}\{\tau > h(x)\} = o(\overline{F}(x))$ for some function $h(x) \to \infty$ such that $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$.

In Section 2, we derive new simple uniform upper bounds for the ratio $\overline{F^{*n}}(x)/\overline{F}(x)$ which generalise Kesten's bound for $\mathscr{S}^*$-distributions. We prove the following

Theorem 2. *Assume that $F \in \mathscr{S}^*$. If $\mathbf{E}\xi < 0$, then there exists a constant K such that*

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq Kn \quad \text{for all } n \text{ and } x.$$

If $\mathbf{E}\xi \in [0, \infty)$, then, for any $c > \mathbf{E}\xi$, there exists K such that

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq \frac{K}{\overline{F}(cn)} \quad \text{for all } n \text{ and } x.$$

The latter estimates are also of their own interest. They substantially improve similar bounds in Shneer [36, Theorems 1 and 2] (see also Daley *et al.* [6, Theorem 3]). In Theorem 4, Section 2, we show that the condition $F \in \mathscr{S}^*$ is essential for the statement of Theorem 2 to hold; more precisely, we construct a distribution $F \in \mathscr{S} \setminus \mathscr{S}^*$ with negative mean such that $\sup_{n,x} \frac{\overline{F^{*n}}(x)}{n\overline{F}(x)} = \infty$.

A closely related topic is the asymptotics of the type $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$ as n, $x \to \infty$ which have been extensively studied starting from 60s. The first works are remarkable papers of S. Nagaev [25, 26], Linnik [21] (in this paper, in a special case, the asymptotics are stated, but the key relation (10.10) on p. 303 is not supported by a proof), and later on of A. Nagaev [23, 24] where in particular the regularly varying distributions were considered. Namely, if F is regularly varying with the parameter $\alpha > 2$ and $\mathbf{E}\xi_1 = 0$, $\mathbf{E}\xi_1^2 = 1$, then under mild technical conditions (see [23], [28, Theorem 1.9], or [32, Theorem 6]) the following asymptotics hold

$$\mathbf{P}\{S_n > x\} \sim \overline{\Phi}(x/\sqrt{n}) + n\overline{F}(x) \quad \text{as } x \to \infty \text{ uniformly in } n \le x^2;$$

here $\overline{\Phi}$ is the tail function of the standard normal law. Further, it follows that, if $x \le \sqrt{(\alpha-2-\varepsilon)n\ln n}$, then the asymptotics follow the Cental Limit Theorem, while if $x > \sqrt{(\alpha-2+\varepsilon)n\ln n}$, then the probability of a single big jump dominates. For Weibull-type distributions the situation is more complicated, see, e.g., A. Nagaev [24], S. Nagaev [27], Rosovskii [33, 34]. Detailed overviews of results in the theory of large deviations for random walks with subexponential increments are given in S. Nagaev [28] and in Mikosch and A. Nagaev [22]. There is still an ongoing research in this area, see recent works by A. Borovkov and K. Borovkov [3], A. Borovkov and Mogulskii [4], Denisov *et al.* [7] and references therein. In Section 3 of this paper, for an *arbitrary* distribution $F \in \mathscr{S}^*$, we find a range for $n = n(x)$ where the asymptotics $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$ hold. The corresponding proof is surprisingly short.

In Section 5, we study the case where the tail distributions of τ and ξ are asymptotically comparable and, for a subclass of subexponential distributions, we obtain the asymptotics for $\mathbf{P}\{S_\tau > x\}$ which differ from (3), see Theorem 8. This generalises results by A. Borovkov and K. Borovkov [3] and by Stam [37], see Section 5 for further comments. As a corollary, in Section 6, we obtain new tail asymptotics for Galton–Watson branching processes.

In Section 7, we study the case where τ may depend on $\{\xi_n\}$, in particular, where τ is a stopping time. First, we prove Theorem 9 where we obtain equivalence (3) for bounded τ. In the proof, we adapt the approach developed in [16] and generalise Greenwood's result onto the whole class of subexponential distributions. Then we consider an unbounded τ and prove Theorem 10 which states that equivalence (3) holds under a stronger assumption than (4) (see condition (37) below). Theorem 10 geleralises earlier results by Greenwood and Monroe [17] and by A. Borovkov and Utev [5], see Corollary 3 and comments after it. Concerning the asymptotics for the maximum, it was shown in [15] (see also [14]) that the equivalence $\mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ holds without any further assumptions on the tail distribution of τ if $\mathbf{E}\xi < 0$ and under condition (37) otherwise.

2. Uniform upper bounds for tails; proof of Theorem 2

In this Section, for the ratios $\overline{F^{*n}}(x)/\overline{F}(x)$, we derive more precise upper bounds than Kesten's bound, which are again uniform in x. We consider two cases $\mathbf{E}\xi < 0$ and $\mathbf{E}\xi \ge 0$ separately. We need the following result:

Theorem 3 ([20] and [8, Corollary 4]). *Assume that $F \in \mathscr{S}^*$ and $\mathbf{E}\xi < 0$. Then, as $x \to \infty$ and uniformly in $n \geq 1$,*

$$\mathbf{P}\{M_n > x\} \sim \frac{1}{|\mathbf{E}\xi|} \int_x^{x+n|\mathbf{E}\xi|} \overline{F}(y)dy.$$

Proof of Theorem 2. First we consider the case (i) of negative mean. Taking into account the inequality $S_n \leq M_n$, Theorem 3, and the inequality

$$\frac{1}{|\mathbf{E}\xi|} \int_x^{x+n|\mathbf{E}\xi|} \overline{F}(y)dy \quad \leq \quad n\overline{F}(x), \tag{5}$$

we obtain statement (i) of the theorem.

Now consider the case (ii) where $\mathbf{E}\xi \geq 0$. Take $c > \mathbf{E}\xi$. Put $\widetilde{\xi}_i = \xi_i - c$ and $\widetilde{S}_n = \widetilde{\xi}_1 + \ldots + \widetilde{\xi}_n$. Then $\mathbf{E}\widetilde{\xi} = \mathbf{E}\xi - c < 0$ and again we can apply Theorem 3. Thus, there exists a constant K_1 such that, for all x and n,

$$\overline{\widetilde{F}^{*n}}(x) \leq K_1 \int_0^{n|\mathbf{E}\widetilde{\xi}|} \overline{\widetilde{F}}(x+y)dy$$

where $\widetilde{F}$ in the distribution of $\widetilde{\xi}$. Therefore,

$$\begin{aligned}\mathbf{P}\{S_n > x\} = \mathbf{P}\{\widetilde{S}_n > x - nc\} \quad &\leq \quad K_1 \int_0^{nc} \overline{\widetilde{F}}(x - nc + y)dy \\ &= \quad K_1 \int_0^{nc} \overline{\widetilde{F}}(x-y)dy.\end{aligned}$$

Since $F \in \mathscr{S}^*$, the distribution F is long-tailed and, hence, $\overline{\widetilde{F}}(x) \sim \overline{F}(x)$ as $x \to \infty$. Then

$$\mathbf{P}\{S_n > x\} \quad \leq \quad K_2 \int_0^{nc} \overline{F}(x-y)dy, \tag{6}$$

for some constant K_2 and for all $x \geq 0$. If $x \geq nc$, then

$$\begin{aligned}\int_0^{nc} \overline{F}(x-y)dy \quad &\leq \quad \int_0^{nc} \overline{F}(x-y)\frac{\overline{F}(y)}{\overline{F}(nc)}dy \\ &\leq \quad \int_0^{x} \overline{F}(x-y)\frac{\overline{F}(y)}{\overline{F}(nc)}dy \leq K_3 \frac{\overline{F}(x)}{\overline{F}(nc)}\end{aligned}$$

where

$$K_3 = \sup_{x \geq 0} \frac{1}{\overline{F}(x)} \int_0^x \overline{F}(x-y)\overline{F}(y)dy$$

is finite, owing to $F \in \mathscr{S}^*$. If $x < nc$, then

$$\overline{F^{*n}}(x) \quad \leq \quad 1 \leq \frac{\overline{F}(x)}{\overline{F}(nc)}.$$

These two bounds together with (6) complete the proof of the second assertion of Theorem 2.

From Theorem 2 and from the dominated convergence theorem, we deduce the following corollary.

Corollary 1. *Tail equivalence* (3) *holds if* $F \in \mathscr{S}^*$ *and* $\mathbf{E}\xi \geq 0$, *provided that*

$$\sum_{n=1}^{\infty} \frac{\mathbf{P}\{\tau = n\}}{\overline{F}(cn)} < \infty \quad \textit{for some } c > \mathbf{E}\xi.$$

The latter condition is stronger than condition (4), because

$$\frac{\mathbf{P}\{\tau > k\}}{\overline{F}(ck)} \leq \sum_{n>k} \frac{\mathbf{P}\{\tau = n\}}{\overline{F}(cn)}.$$

Now let us discuss the importance of the condition $F \in \mathscr{S}^*$ in Theorem 2. The following observation shows the essence of the difference between two classes of distributions, $\mathscr{S}$ and $\mathscr{S}^*$, is the following one. Let a long-tailed distribution F be absolutely continious with density f. For any function $h(x) > 0$,

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)F(dy) = \int_{h(x)}^{x-h(x)} \overline{F}(x-y)f(y)dy.$$

Then F is subexponential if and only if

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)f(y)dy = o(\overline{F}(x)) \text{ as } x \to \infty$$

holds for any function $h(x) \to \infty$; equivalently, if it holds for some function $h(x) \to \infty$ such that $\overline{F}(x - h(x)) \sim \overline{F}(x)$. On the other hand, $F \in \mathscr{S}^*$ if and only if

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)\overline{F}(y)dy = o(\overline{F}(x)) \text{ as } x \to \infty.$$

In typical cases $f(x) = o(\overline{F}(x))$ and, hence,

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)f(y)dy = o\left(\int_{h(x)}^{x-h(x)} \overline{F}(x-y)\overline{F}(y)dy\right) \text{ as } x \to \infty.$$

It means that the subexponentiality of F is more likely than $F \in \mathscr{S}^*$. The latter observation gives the idea how to show that the condition $F \in \mathscr{S}^*$ in Theorem 2 cannot be extended to the subexponentiality of F.

Theorem 4. *There exists a subexponential distribution* F *on* $\mathbf{R}$ *with a negative mean such that*

$$\overline{F^{*n_k}}(x_k) \geq c\frac{n_k^2}{\ln n_k}\overline{F}(x_k),$$

for some $c > 0$ *and for some sequences* n_k, $x_k \to \infty$.

The latter theorem yields that, for some distribution $F \in \mathscr{S} \setminus \mathscr{S}^*$ with negative mean, the first estimate of Theorem 2 fails, that is, $\sup_{n,x} \frac{\overline{F^{*n}}(x)}{n\overline{F}(x)} = \infty$.

Proof of Theorem 4. We start with a construction of a specific subexponential distribution G on the positive half-line. Put $R_0 = 0$, $R_1 = 1$ and $R_{k+1} = e^{R_k}/R_k$ for $k \ge 1$. Since e^x/x is increasing for $x \ge 1$, the sequence R_k is increasing and

$$R_k = o(R_{k+1}) \text{ as } k \to \infty. \tag{7}$$

Put $t_k = R_k^2$. Define the hazard function $R(x) \equiv -\ln \overline{G}(x)$ as

$$R(x) = R_k + r_k(x - t_k) \quad \text{for } x \in (t_k, t_{k+1}],$$

where

$$r_k = \frac{R_{k+1} - R_k}{t_{k+1} - t_k} = \frac{1}{R_{k+1} + R_k} \sim \frac{1}{R_{k+1}} \tag{8}$$

by (7). In other words, the hazard rate $r(x) = R'(x)$ is defined as $r(x) = r_k$ for $x \in (t_k, t_{k+1}]$, where r_k is given by (8). By the construction, we have $\overline{G}(t_k) = e^{-\sqrt{t_k}}$, so that at points t_k the tail of G behaves like the Weibull tail with parameter $1/2$. Between these points the tail decays exponentially with indexes r_k.

We prove now that G has finite mean and is subexponential. Since by (8)

$$\begin{aligned}\int_{t_k}^{t_{k+1}} e^{-R(y)} dy &= r_k^{-1}(e^{-R_k} - e^{-R_{k+1}}) \\ &\sim r_k^{-1} e^{-R_k} \sim R_{k+1} e^{-R_k} = 1/R_k,\end{aligned}$$

the mean of G,

$$\int_0^\infty \overline{G}(y) dy = \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} \overline{G}(y) dy,$$

is finite.

It follows from the definition that $r(x)$ decreases to 0. Then we can apply Pitman's criterion [31] which says that G is subexponential if the function $e^{yr(y)-R(y)} r(y)$ is integrable over $[0, \infty)$. In order to estimate the integral of this function, put

$$I_k = \int_{t_k}^{t_{k+1}} e^{yr(y)-R(y)} r(y) dy.$$

Then

$$I_k = r_k \int_{t_k}^{t_{k+1}} e^{yr_k - (R_k + r_k(y - t_k))} dy \le r_k e^{-R_k + r_k t_k} t_{k+1}.$$

Since

$$r_k t_{k+1} = r_k R_{k+1}^2 \sim R_{k+1} \tag{9}$$

by (8) and

$$r_k t_k = r_k R_k^2 \sim R_k^2 / R_{k+1} = R_k^3 e^{-R_k} \to 0,$$

we get $I_k \le 2R_{k+1}e^{-R_k} \sim 2/R_k$ for k sufficiently large. Therefore,

$$\int_0^\infty e^{yr(y)-R(y)}r(y)dy = \sum_{k=0}^\infty I_k < \infty,$$

and G is indeed subexponential.

In the sequel we need to know the asymptotic behaviour of the following internal part of the convolution integral at point t_k:

$$J_k = \int_{t_k/4}^{3t_k/4} \overline{G}(t_k - y)G(dy) = \int_{t_k/4}^{3t_k/4} e^{-R(t_k-y)}e^{-R(y)}r(y)dy.$$

Owing to (7), $t_{k-1} = o(t_k)$. Thus, $(t_k/4, 3t_k/4] \subset (t_{k-1}, t_k - t_{k-1}]$ for all sufficiently large k. For those values of k, we have

$$\begin{aligned} J_k &= \overline{G}(t_k)\int_{t_k/4}^{3t_k/4} e^{-(-r_{k-1}y)}e^{-(R_{k-1}+r_{k-1}(y-t_{k-1}))}r_{k-1}dy \\ &\ge \overline{G}(t_k)(t_k/2)e^{-R_{k-1}}r_{k-1}. \end{aligned}$$

Applying (9) and the equality $e^{R_{k-1}} = R_kR_{k-1}$, we obtain, for all sufficiently large k,

$$J_k \ge \overline{G}(t_k)e^{-R_{k-1}}R_k/3 = \overline{G}(t_k)/3R_{k-1}. \tag{10}$$

Let η_1, η_2, ... be independent random variables with common distribution G and put $T_n = \eta_1 + \ldots + \eta_n$. For any n, we have

$$\begin{aligned} \mathbf{P}\{T_n > x\} &\ge \sum_{1\le i<j\le n} \mathbf{P}\{T_n > x, \eta_i > n, \eta_j > n, \eta_l \le n \text{ for all } l \ne i, j\} \\ &= \frac{n(n-1)}{2}\mathbf{P}\{T_n > x, \eta_1 > n, \eta_2 > n, \eta_3 \le n, \ldots, \eta_n \le n\}. \end{aligned}$$

Since η's are positive, the latter probability is not smaller than

$$\mathbf{P}\{\eta_1 + \eta_2 > x, \eta_1 > n, \eta_2 > n\}\mathbf{P}\{\eta_3 \le n, \ldots, \eta_n \le n\}.$$

The mean of η is finite, thus $\overline{G}(n) = o(1/n)$ as $n \to \infty$ and

$$\mathbf{P}\{\eta_3 \le n, \ldots, \eta_n \le n\} = (1 - \overline{G}(n))^{n-2} \to 1.$$

Putting altogether, we get, for all sufficiently large n, the following estimate from below

$$\mathbf{P}\{T_n > x\} \ge \frac{n^2}{3}\mathbf{P}\{\eta_1 + \eta_2 > x, \eta_1 > n, \eta_2 > n\}. \tag{11}$$

Now take $n = n_k = [\sqrt{t_k}] = [R_k]$. Then, for all sufficiently large k (at least for those k where $n_k < t_k/4$),

$$\mathbf{P}\{\eta_1 + \eta_2 > t_k, \eta_1 > n_k, \eta_2 > n_k\} \ge J_k.$$

Therefore, by (11) and (10), for all sufficiently large k,

$$\mathbf{P}\{T_{n_k} > t_k\} \geq n_k^2 \overline{G}(t_k)/9R_{k-1} \sim n_k^2 \overline{G}(t_k)/9 \ln n_k,$$

due to $R_{k-1} \sim \ln R_k \sim \ln n_k$.

Denote $b = \mathbf{E}\eta_1$. Put $\xi_i = \eta_i - 2b$, then ξ's have negative mean and $S_n = T_n - 2nb$. Denote by F the distribution of ξ_1; it is subexponential because G is.

Take $x = x_k = t_k - 2n_k b$, so that $x_k \sim n_k^2$. By the latter inequality we have

$$\mathbf{P}\{S_{n_k} > x_k\} = \mathbf{P}\{T_{n_k} > t_k\} \geq n_k^2 \overline{G}(t_k)/10 \ln n_k.$$

Note also that

$$\overline{F}(x_k) = \overline{G}(t_k - 2n_k b) = \overline{G}(t_k) e^{r_{k-1} 2n_k b} \leq \overline{G}(t_k) e^{2b}$$

because $r_{k-1} n_k \leq r_{k-1} R_k \leq 1$ by (8). Therefore, the inequality

$$\mathbf{P}\{S_{n_k} > x_k\} \geq n_k^2 \overline{F}(x_k) e^{-2b}/10 \ln n_k$$

holds which yields the conclusion of the theorem.

The subexponential distribution G constructed in the latter proof cannot belong to the class $\mathscr{S}^*$ because otherwise the theorem conclusion fails, as follows from Theorem 2. The fact that $G \notin \mathscr{S}^*$ can also be proved directly. Klüppelberg's criterion [18] states that $G \in \mathscr{S}^*$ if and only if

$$\int_0^x e^{yr(x)-R(y)} dy \quad \to \quad \int_0^\infty \overline{G}(y) dy \quad \text{ as } x \to \infty.$$

In our construction,

$$\begin{aligned} \int_0^{t_k-0} e^{yr(t_k-0)-R(y)} dy &\geq \int_{t_{k-1}}^{t_k} e^{yr_{k-1}-R(y)} dy \\ &\geq (t_k - t_{k-1}) e^{-R_{k-1}} \\ &\sim R_k^2 e^{-R_{k-1}} = e^{R_{k-1}}/R_{k-1}^2 \to \infty \end{aligned}$$

as $k \to \infty$. Hence, $G \notin \mathscr{S}^*$.

3. On the asymptotics $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$

As before, we assume $\mathbf{E}\xi$ to be finite. Then, by the Strong Law of Large Numbers,

$$\mathbf{P}\{S_n > -An\} \to 1 \text{ as } A \to \infty \text{ uniformly in } n \geq 1, \tag{12}$$

and by the Chebyshev's inequality

$$\mathbf{P}\{\xi_1 > An\} \leq \mathbf{E}|\xi_1|/An \text{ for all } A > 0 \text{ and } n \geq 1. \tag{13}$$

Theorem 5. *Let $F \in \mathscr{S}^*$ and let an increasing function $h(x) > 0$ be such that $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$. Then $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$ as $x \to \infty$ uniformly in $n \leq h(x)$.*

Proof of the lower bound is similar to that in [7, Section 4]. Fix $A > 0$. We use the following inequalities:

$$\begin{aligned}
\mathbf{P}\{S_n > x\} &\geq \sum_{i=1}^{n} \mathbf{P}\{S_n > x, \xi_i > x + An, \xi_j \leq An \text{ for all } j \neq i\} \\
&\geq n\mathbf{P}\{S_n - \xi_1 > -An, \xi_1 > x + An, \xi_2 \leq An, \ldots, \xi_n \leq An\} \\
&= n\overline{F}(x + An)\mathbf{P}\{S_{n-1} > -An, \xi_1 \leq An, \ldots, \xi_{n-1} \leq An\}.
\end{aligned}$$

We have $\overline{F}(x + An) \sim \overline{F}(x)$ as $x \to \infty$ uniformly in $n \leq h(x)$. Taking also into account that

$$\mathbf{P}\{S_{n-1} > -An, \xi_1 \leq An, \ldots, \xi_{n-1} \leq An\} \quad \geq \quad \mathbf{P}\{S_{n-1} > -An\} - (n-1)\mathbf{P}\{\xi_1 > An\},$$

we get, for any fixed $A > 0$,

$$\liminf_{x\to\infty} \inf_{n \leq h(x)} \frac{\mathbf{P}\{S_n > x\}}{n\overline{F}(x)} \quad \geq \quad \inf_n (\mathbf{P}\{S_{n-1} > -An\} - (n-1)\mathbf{P}\{\xi_1 > An\}).$$

Since the infimum on the right goes to 1 as $A \to \infty$ owing to (12) and (13), we arrive at the following lower bound

$$\liminf_{x\to\infty} \inf_{n \leq h(x)} \frac{\mathbf{P}\{S_n > x\}}{n\overline{F}(x)} \quad \geq \quad 1.$$

To prove the upper bound, we apply Theorem 3 to random variables $\widetilde{\xi}_i = \xi_i - \mathbf{E}\xi_1 - 1$ with negative mean $\mathbf{E}\widetilde{\xi}_i = -1$ and to $\widetilde{S}_n = S_n - n(\mathbf{E}\xi_1 + 1)$. Thus,

$$\begin{aligned}
\mathbf{P}\{S_n > x\} &= \mathbf{P}\{\widetilde{S}_n > x - n(\mathbf{E}\xi_1 + 1)\} \\
&\leq (1 + o(1)) \int_{x - n(\mathbf{E}\xi_1 + 1)}^{x - n\mathbf{E}\xi_1} \overline{\widetilde{F}}(x + u)du \\
&\leq (1 + o(1))n\overline{\widetilde{F}}(x - n(\mathbf{E}\xi_1 + 1))
\end{aligned}$$

as $x \to \infty$ where $\widetilde{F}$ is the distribution of $\widetilde{\xi}$. If $n \leq h(x)$ then $\overline{\widetilde{F}}(x - n(\mathbf{E}\xi_1 + 1)) \sim \overline{F}(x)$ as $x \to \infty$ and the proof is complete.

The range $n \leq h(x)$ is usually more narrow than one could expect. Say, for the regularly varying distributions (more generally, for the intermediate regularly varying, see the definition in Section 5) we can take $h(x) = o(x)$. Then we get the range $n = o(x)$ while the standard (if the mean is zero and the second moment is finite) range is $x^2 > cn \ln n$; in the class of distributions with finite mean, the relation $\mathbf{P}\{S_n > x\} \sim n\overline{F}(x)$ holds in the range $x > (\mathbf{E}\xi + \varepsilon)n$, $\varepsilon > 0$, see S. Nagaev [29]. The advantage of the result in Theorem 5 is its simplicity and universality since it is valid for all distributions from $\mathscr{S}^*$ without any further moment or regularity assumptions, compare with a series of results in [3, 4, 7] where the hazard rate is assumed to be sufficiently smooth.

As follows from [7], if the mean is zero and the second moment is finite, then the right range should be $n \leq h^2(x)$, roughly speaking. Our technique allows to prove the lower bound for this range.

Theorem 6. *Let $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 < \infty$. Let F be a long-tailed distribution and let an increasing function $h(x) > 0$ be such that $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$. Then $\mathbf{P}\{S_n > x\} \geq (1 + o(1))n\overline{F}(x)$ as $x \to \infty$ uniformly in $n \leq h^2(x)$.*

Proof. Fix $A > 0$. By the Chebyshev's inequality,

$$\mathbf{P}\{\xi_1 > A\sqrt{n}\} \le \mathbf{E}\xi^2/A^2 n \ \text{ and } \ \mathbf{P}\{S_n > -A\sqrt{n}\} \ge 1 - \mathbf{E}\xi^2/A^2. \tag{14}$$

In this proof we use a slightly different inequality than in the previous theorem:

$$\begin{aligned}\mathbf{P}\{S_n > x\} &\ge \sum_{i=1}^{n} \mathbf{P}\{S_n > x, \xi_i > x + A\sqrt{n}, \xi_j \le A\sqrt{n} \text{ for all } j \ne i\} \\ &\ge n\mathbf{P}\{S_n - \xi_1 > -A\sqrt{n}, \xi_1 > x + A\sqrt{n}, \xi_2 \le A\sqrt{n}, \ldots, \xi_n \le A\sqrt{n}\} \\ &= n\overline{F}(x + A\sqrt{n})\mathbf{P}\{S_{n-1} > -A\sqrt{n}, \xi_1 \le A\sqrt{n}, \ldots, \xi_{n-1} \le A\sqrt{n}\}.\end{aligned}$$

Since $n \le h^2(x)$, $\overline{F}(x + A\sqrt{n}) \sim \overline{F}(x)$ as $x \to \infty$. Applying (14), we get

$$\begin{aligned}&\mathbf{P}\{S_{n-1} > -A\sqrt{n}, \xi_1 \le A\sqrt{n}, \ldots, \xi_{n-1} \le A\sqrt{n}\} \\ &\quad\ge \mathbf{P}\{S_{n-1} > -A\sqrt{n}\} - (n-1)\mathbf{P}\{\xi_1 > A\sqrt{n}\} \\ &\quad\ge 1 - 2\mathbf{E}\xi^2/A^2 \to 1 \ \text{ as } A \to \infty.\end{aligned}$$

Now the lower bound for $\mathbf{P}\{S_n > x\}$ follows.

4. Proof of Theorem 1

Since τ is independent of ξ's, we can use the following decomposition:

$$\mathbf{P}\{S_\tau > x\} = \sum_{n=0}^{\infty} \mathbf{P}\{\tau = n\}\overline{F^{*n}}(x).$$

By the subexponentiality, here the nth term is equivalent to $n\mathbf{P}\{\tau = n\}\overline{F}(x)$ as $x \to \infty$. In particular, by Fatou's lemma,

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \ge \sum_{n=0}^{\infty} n\mathbf{P}\{\tau = n\} = \mathbf{E}\tau, \tag{15}$$

without any condition on the sign of $\mathbf{E}\xi$. In the case of negative mean, the nth term is bounded from above by $n\overline{F}(x)$, see (5). Then the dominated convergence for series yields statement (i) of the theorem.

Now turn to the proof of statement (ii) where $\mathbf{E}\xi \ge 0$. Since $S_\tau \le M_\tau$, it follows from (15) that it is sufficient to prove that

$$\mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x) \ \text{ as } x \to \infty. \tag{16}$$

To prove the latter relation, we start with the following representation: for any N,

$$\begin{aligned}\mathbf{P}\{M_\tau > x\} &= \mathbf{P}\{M_\tau > x, \tau \le N\} + \mathbf{P}\{M_\tau > x, \tau \in (N, x/c]\} + \mathbf{P}\{M_\tau > x, c\tau > x\} \\ &\equiv P_1 + P_2 + P_3.\end{aligned} \tag{17}$$

Since any $\mathscr{S}^*$-distribution is subexponential and $S_n \le M_n \le \xi_1^+ + \ldots + \xi_n^+$,

$$\mathbf{P}\{M_n > x\} \sim n\overline{F}(x)$$

as $x \to \infty$, for any n. Thus, for any fixed N,

$$\mathbf{P}\{M_\tau > x, \tau \le N\} = \sum_{n=1}^{N} \mathbf{P}\{\tau = n\}\mathbf{P}\{M_n > x\} \sim \mathbf{E}\{\tau; \tau \le N\}\overline{F}(x)$$

as $x \to \infty$ which implies the existence of an increasing function $N(x) \to \infty$ such that

$$P_1 = \mathbf{P}\{M_\tau > x, \tau \le N(x)\} \sim \mathbf{E}\tau\overline{F}(x). \tag{18}$$

In what follows, we use representation (17) with $N(x)$ in place of N. We further estimate the second term on the right side in (17). Let $\varepsilon = (c - \mathbf{E}\xi)/2 > 0$ and let $b = (\mathbf{E}\xi + c)/2$. Consider $\widetilde{\xi}_n = \xi_n - b$, $\widetilde{S}_n = \widetilde{\xi}_1 + \ldots + \widetilde{\xi}_n$ and $\widetilde{M}_n = \max(\widetilde{S}_1, \ldots, \widetilde{S}_n)$. Then $\mathbf{E}\widetilde{\xi} = -\varepsilon < 0$ and we can apply Theorem 3. Taking into account that $M_n \le \widetilde{M}_n + bn$, we obtain that there exists K such that, for all x and n,

$$\begin{aligned} \mathbf{P}\{M_n > x\} &\le \mathbf{P}\{\widetilde{M}_n > x - bn\} \\ &\le K \int_0^{n\varepsilon} \overline{\widetilde{F}}(x - nb + y)dy \\ &\le K \int_0^{n\varepsilon} \overline{F}(x - nb + y)dy. \end{aligned}$$

Hence,

$$P_2 = \mathbf{P}\{M_\tau > x, \tau \in (N(x), x/c]\} \le K \sum_{n=N(x)}^{[x/c]} \mathbf{P}\{\tau = n\} \int_0^{n\varepsilon} \overline{F}(x - nb + y)dy.$$

Since $b - \varepsilon = \mathbf{E}\xi$,

$$\int_0^{n\varepsilon} \overline{F}(x - nb + y)dy = \int_{n\mathbf{E}\xi}^{nb} \overline{F}(x - y)dy.$$

Then

$$\begin{aligned} P_2 &\le K \int_{N(x)\mathbf{E}\xi}^{b[x/c]} \overline{F}(x - y)dy \sum_{n=\max(N(x),[y/b]+1)}^{[x/c]} \mathbf{P}\{\tau = n\} \\ &\le K \int_{N(x)\mathbf{E}\xi}^{bx/c} \overline{F}(x - y)\mathbf{P}\{\tau > y/b\}dy \\ &\le K \int_{N(x)\mathbf{E}\xi}^{bx/c} \overline{F}(x - y)\mathbf{P}\{\tau > y/c\}dy, \end{aligned} \tag{19}$$

because $b < c$. By condition (4), $\mathbf{P}\{\tau > y/c\} \le K_1\overline{F}(y)$, for some K_1 and all y. Therefore, the inequality

$$P_2 \le KK_1 \int_{N(x)\mathbf{E}\xi}^{bx/c} \overline{F}(x - y)\overline{F}(y)dy = o(\overline{F}(x)) \text{ as } x \to \infty \tag{20}$$

follows from $b/c < 1$ and from $F \in \mathscr{S}^*$. Indeed, for any $\mathscr{S}^*$-distribution,

$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)\overline{F}(y)dy = o(\overline{F}(x)) \text{ as } x \to \infty, \tag{21}$$

for any function $h(x) \to \infty$ such that $h(x) \le x/2$ (see, e.g., [18]).

Now we estimate the third term on the right in (17) using condition (4):

$$P_3 \le \mathbf{P}\{c\tau > x\} = o(\overline{F}(x)) \text{ as } x \to \infty. \tag{22}$$

Altogether relations (18), (20), and (22) complete the proof of Theorem 1.

Now we provide an example where

$$\frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \to \infty$$

given that condition (4) is satisfied only with $c = \mathbf{E}\xi > 0$ and not with any bigger c. Assume that F is a Weibull distribution on the positive half line with parameter $\beta \in (1/2, 1)$, that is $\overline{F}(x) = e^{-x^\beta}$. Let τ have a distribution such that $\mathbf{P}\{c\tau > x\} \sim x^{-1}e^{-x^\beta}$ as $x \to \infty$. Write down the following lower bound:

$$\mathbf{P}\{S_\tau > x\} \quad \ge \quad \mathbf{P}\{S_\tau > x | c\tau > x - \sqrt{x}\}\mathbf{P}\{c\tau > x - \sqrt{x}\}.$$

By the Central Limit Theorem,

$$\delta \equiv \liminf_{x\to\infty} \mathbf{P}\{S_\tau > x | c\tau > x - \sqrt{x}\} \ge \liminf_{x\to\infty} \mathbf{P}\{S_{[(x-\sqrt{x})/c]} > x\} > 0.$$

Hence,

$$\begin{aligned} \liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \quad &\ge \quad \delta \liminf_{x\to\infty} \frac{\mathbf{P}\{c\tau > x - \sqrt{x}\}}{\overline{F}(x)} \\ &= \quad \delta \liminf_{x\to\infty} \frac{e^{x^\beta - (x-\sqrt{x})^\beta}}{x - \sqrt{x}} = \infty, \end{aligned}$$

because $\beta > 1/2$.

We conclude this section by an example showing that the conclusion of Theorem 1 cannot hold for all subexponential distributions. Indeed, take F with negative mean as described in Theorem 4. Without loss of generality we assume that the series $\sum_k n_k^{-1} \ln n_k$ converge. Consider τ taking values n_k with probabilities $c \ln^2 n_k / n_k^2$, here c is the normalising constant. Then τ has a finite mean, but

$$\mathbf{P}\{S_\tau > x_k\} \ge \mathbf{P}\{S_{n_k} > x_k\}\mathbf{P}\{\tau = n_k\} \ge c\frac{n_k^2}{\ln n_k}\overline{F}(x_k)\frac{\ln^2 n_k}{n_k^2},$$

so that, as $k \to \infty$,

$$\frac{\mathbf{P}\{S_\tau > x_k\}}{\overline{F}(x_k)} \to \infty.$$

5. The case where ξ and τ may be tail-comparable

In this section we do not assume condition (4) to hold, such a situation is of particular importance for branching processes. To start with, we define two important classes of distributions.

A distribution F is called *dominated varying* if there exists c such that $\overline{F}(x) \le c\overline{F}(2x)$ for all x. It is known that any long-tailed and dominated varying distribution with a finite mean belongs to the class $\mathscr{S}^*$, see [18].

We say that a distribution G is *intermediate regularly varying* (at infinity) if

$$\lim_{\varepsilon\downarrow 0} \limsup_{x\to\infty} \frac{\overline{G}((1-\varepsilon)x)}{\overline{G}(x)} = 1. \tag{23}$$

In particular, any regularly varying at infinity distribution satisfies the latter relation. Any intermediate regularly varying distribution is long-tailed and dominated varying; in particular, it belongs to the class $\mathscr{S}^*$, provided its mean is finite.

Theorem 7. *Let $F \in \mathscr{S}^*$, $\mathbf{E}\xi > 0$, and*

$$\overline{F}(x) = O(\mathbf{P}\{\tau > x\}) \text{ as } x \to \infty. \tag{24}$$

If the distribution of τ is intermediate regularly varying, then

$$\mathbf{P}\{S_\tau > x\} \sim \mathbf{P}\{M_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x) + \mathbf{P}\{\tau > x/\mathbf{E}\xi\} \text{ as } x \to \infty. \tag{25}$$

We strongly believe that the statement of the theorem stays valid in a more general setting where the distribution of τ is assumed to be *square-root insensitive*, that is $\mathbf{P}\{\tau > x \pm \sqrt{x}\} \sim \mathbf{P}\{\tau > x\}$, and the variance of ξ is finite. Probably, some further minor regularity assumptions are required. For example, the Weibull distribution $\overline{F}(x) = e^{-x^\beta}$ with parameter $\beta < 1/2$ is square-root insensitive. For distribution which is *not* square-root insensitive, the asymptotics are different and more complicated.

Proof of Theorem 7. By (23), for any fixed $\delta > 0$, we can choose $a < \mathbf{E}\xi$ and $c > \mathbf{E}\xi$ sufficiently close to $\mathbf{E}\xi$ such that

$$1 - \delta/2 \le \liminf_{x\to\infty} \frac{\mathbf{P}\{a\tau > x\}}{\mathbf{P}\{\tau > x/\mathbf{E}\xi\}} \le \limsup_{x\to\infty} \frac{\mathbf{P}\{c\tau > x\}}{\mathbf{P}\{\tau > x/\mathbf{E}\xi\}} \le 1 + \delta/2.$$

Then, due to $S_\tau \le M_\tau$, it is sufficient to prove the following lower bound for the sum

$$\mathbf{P}\{S_\tau > x\} \quad \ge \quad (\mathbf{E}\tau + o(1))\overline{F}(x) + (1 + o(1))\mathbf{P}\{\tau > x/a\}. \tag{26}$$

and the upper bound for the maximum

$$\mathbf{P}\{M_\tau > x\} \quad \le \quad (\mathbf{E}\tau + o(1))\overline{F}(x) + (1 + o(1))\mathbf{P}\{\tau > x/c\} \text{ as } x \to \infty. \tag{27}$$

We have

$$\mathbf{P}\{S_\tau > x\} \quad = \quad \mathbf{P}\{S_\tau > x, \tau \le x/a\} + \mathbf{P}\{S_\tau > x, \tau > x/a\}.$$

Since $a < \mathbf{E}\xi$, $\mathbf{P}\{S_\tau > x|\tau > x/a\} \to 1$ as $x \to \infty$, by the Law of Large Numbers. Now the standard arguments lead to (26).

To prove the upper bound, we use a representation similar to (17) (see the previous proof):

$$\begin{aligned}\mathbf{P}\{M_\tau > x\} &= \mathbf{P}\{M_\tau > x, \tau \le N(x)\} + \mathbf{P}\{M_\tau > x, \tau \in (N(x), x/c]\} + \mathbf{P}\{M_\tau > x, c\tau > x\} \\ &\equiv P_1 + P_2 + P_3.\end{aligned}$$

The first summand P_1 can be treated as ealier. The second summand P_2 can be estimated as follows: if condition (24) holds then, by estimate (19),

$$P_2 \le KK_2 \int_{N(x)\mathbf{E}\xi}^{bx/c} \mathbf{P}\{\tau > x-y\}\mathbf{P}\{\tau > y\}dy,$$

for some K_2. Since the distribution of τ is intermediate regularly varying and, therefore, belongs to $\mathscr{S}^*$,

$$P_2 = o(\mathbf{P}\{\tau > x\}).$$

Taking into account also that $P_3 \le \mathbf{P}\{c\tau > x\}$, we finally get

$$\mathbf{P}\{M_\tau > x\} \le (\mathbf{E}\tau + o(1))\overline{F}(x) + \mathbf{P}\{\tau > x/c\} + o(\mathbf{P}\{\tau > x\}) \text{ as } x \to \infty.$$

Since the distribution of τ is (in particular) dominated varying, $\mathbf{P}\{\tau > x\} = O(\mathbf{P}\{\tau > x/c\})$. Therefore, (27) is proved and the conclusion of Theorem 7 follows.

Theorem 8. *Let $\mathbf{E}\xi > 0$ and let τ have an intermediate regularly varying distribution. If the distribution F is long-tailed and dominated varying, then* (25) *holds.*

A particular corollary is that if both ξ and τ have regularly varying tail distributions, then asymptotics (25) hold; this result was proved by Stam [37, Theorems 1.3 and 1.4] for positive ξ and by A. Borovkov and K. Borovkov [3, Section 7.1] for signed ξ. Also, Theorems 7 and 8 generalise and improve Theorem 1.3 of Aleškevičenė *et al.* [1].

Proof of Theorem 8. It follows the lines of the previous proof, and only the term P_2 needs a different estimation. From bound (19), we get

$$P_2 \le K\overline{F}(x - bx/c) \int_{N(x)\mathbf{E}\xi}^{bx/c} \mathbf{P}\{c\tau > y\}dy.$$

Since F is dominated varying, $\overline{F}(x - bx/c) = O(\overline{F}(x))$ as $x \to \infty$. Therefore, $P_2 = o(\overline{F}(x))$ and the proof is complete.

6. Applications to the branching processes

A Galton–Watson process is a stochastic process $\{X_n\}$ which evolves according to the recurrence formula $X_0 = 1$ and

$$X_{n+1} = \sum_{j=1}^{X_n} \xi_j^{(n+1)},$$

where $\{\xi_j^{(n)}\}$ is a family of independent identically distributed non-negative integer-valued random variables with a finite mean, and their common distribution does not depend on n. Here X_n is the number of items in the nth generation. Taking into account that any intermediate regularly varying distribution with finite mean belongs to the class $\mathscr{S}^*$, we obtain the following application of Theorem 7 to the branching process:

Corollary 2. *Let the common distribution of ξ's be intermediate regularly varying. Then, as $x \to \infty$,*

$$\mathbf{P}\{X_2 > x\} \sim \mathbf{E}\xi \mathbf{P}\{\xi > x\} + \mathbf{P}\{\xi > x/\mathbf{E}\xi\}.$$

In particular, if the branching process is *critical*, i.e. if $\mathbf{E}\xi = 1$, then

$$\mathbf{P}\{X_2 > x\} \sim 2\mathbf{P}\{\xi > x\} \text{ as } x \to \infty.$$

More generally, by induction arguments, the tail of the distribution of the number of items in the nth generation is asymptotically equivalent to $n\mathbf{P}\{\xi > x\}$. A similar result (for critical process) was obtained in [38, Theorem 2] in the case of regularly varying distribution of ξ's and for possibly growing n.

7. Equivalences in the case where a counting random variable τ may depend on ξ's

We continue to assume that random variables $\{\xi_n\}$ are independent and identically distributed. For any family Ξ of random variables, denote by $\sigma(\Xi)$ the σ-algebra generated by Ξ. Traditionally, a counting random variable τ is called a *stopping time* for a sequence $\{\xi_n\}$ if $\{\tau \le n\} \in \sigma(\xi_1, \ldots, \xi_n)$ for all n.

We say that a counting random variable τ *does not depend on the future of the sequence* $\{\xi_n\}$ if the family $(\xi_1, \ldots, \xi_n, \mathbf{I}\{\tau \le n\})$ does not depend on $(\xi_j, j \ge n+1)$ for all n. Dependence of this type goes back to Kolmogorov and Prokhorov [19] who proved Wald's identity under the condition that the event $\{\tau \le n\}$ does not depend on ξ_j for all $n \ge 1$ and $j \ge n+1$.

Provided independence of ξ's, any stopping time τ does not depend on the future of the sequence $\{\xi_n\}$. If a counting random variable τ does not depend on ξ's, then it does not depend on the future of the sequence $\{\xi_n\}$.

Let $\mathscr{F}_n$ be a filtration of σ-algebras. A counting random variable τ is called a stopping time for this filtration if $\{\tau \le n\} \in \mathscr{F}_n$ for all n. In this terminology, τ is a stopping time for a sequence $\{\xi_n\}$ if and only if τ is a stopping time for the natural filtration $\mathscr{F}_n = \sigma(\xi_1, \ldots, \xi_n)$.

Consider a special filtration $\mathscr{F}_n = \sigma(\xi_k, \mathbf{I}\{\tau = k\}, k \le n)$. Then τ is a stopping time for this filtration. In addition, τ does not depend on the future of the sequence $\{\xi_n\}$ if and only if $(\xi_j, j \ge n+1)$ does not depend on $\mathscr{F}_n$ for all n.

We start with a result for a bounded counting stopping time (recall that a random variable is *bounded* if its distribution has a bounded support).

Theorem 9. *Let ξ have a subexponential distribution F on $\mathbf{R}$ (we do not assume finite mean), and let the counting variable τ do not depend on the future. If τ is bounded, then $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ as $x \to \infty$.*

Similar result for M_τ may be found in [14, Theorem 1]. Note that one cannot expect the latter asymptotics to hold for any τ with unbounded support, which may depend on $\{\xi_n\}$ – even for a stopping time. Indeed, consider a stopping time $\tau = \min\{n : S_n \le 0\}$. If $\mathbf{E}\xi < 0$ then $\mathbf{E}\tau$ is finite but $\mathbf{P}\{S_\tau > x\} = 0$ for any $x > 0$.

Proof. We adopt the corresponding proof from Greenwood [16] where a stopping time and regularly varying tails were considered. Let N be such that $\mathbf{P}\{\tau \le N\} = 1$. The starting point of the proof is the following representation:

$$\begin{aligned}\mathbf{P}\{S_\tau > x\} &= \sum_{n=1}^{N}(\mathbf{P}\{S_n > x, \tau \geq n\} - \mathbf{P}\{S_n > x, \tau \geq n+1\}) \\ &= \mathbf{P}\{S_1 > x, \tau \geq 1\} + \sum_{n=2}^{N}(\mathbf{P}\{S_n > x, \tau \geq n\} - \mathbf{P}\{S_{n-1} > x, \tau \geq n\}).\end{aligned}$$

Therefore,

$$\mathbf{P}\{S_\tau > x\} = \overline{F}(x) + \sum_{n=2}^{N}(\mathbf{P}\{S_{n-1} \leq x, S_n > x, \tau \geq n\} - \mathbf{P}\{S_{n-1} > x, S_n \leq x, \tau \geq n\}).$$

Now it suffices to show that, for each n,

$$P_1 \equiv \mathbf{P}\{S_{n-1} \leq x, S_n > x, \tau \geq n\} \sim \overline{F}(x)\mathbf{P}\{\tau \geq n\} \tag{28}$$

and

$$P_2 \equiv \mathbf{P}\{S_{n-1} > x, S_n \leq x, \tau \geq n\} = o(\overline{F}(x)). \tag{29}$$

The subexponentiality of F implies that, for each $n \geq 2$,

$$\mathbf{P}\{S_n > x\} \sim n\overline{F}(x) \quad \text{as } x \to \infty. \tag{30}$$

In particular, there exists c such that, for all $n = 2, \ldots, N$,

$$\mathbf{P}\{S_n > x\} \leq c\overline{F}(x) \quad \text{for all } x. \tag{31}$$

The subexponentiality of F also implies, for any $A(x) \to \infty$ such that $\overline{F}(x + A(x)) \sim \overline{F}(x)$,

$$\int_{A(x)}^{x+A(x)} \overline{F}(x-y)F(dy) = o(\overline{F}(x)) \quad \text{as } x \to \infty. \tag{32}$$

To establish (28), we first note that $\{\tau \geq n\} = \overline{\{\tau \leq n-1\}}$ and thus $\sigma(S_{n-1}, \mathbf{I}\{\tau \geq n\})$ does not depend on ξ_n, since τ does not depend on the future. This implies

$$\begin{aligned}P_1 &= \int_0^\infty \mathbf{P}\{S_{n-1} \in (x-y, x], \xi_n \in dy, \tau \geq n\} \\ &= \int_0^\infty \mathbf{P}\{S_{n-1} \in (x-y, x], \tau \geq n\}F(dy).\end{aligned}$$

We use the following decomposition, $A > 0$:

$$\begin{aligned}P_1 &= \left(\int_0^A + \int_A^{x+A} + \int_{x+A}^\infty\right)\mathbf{P}\{S_{n-1} \in (x-y, x], \tau \geq n\}F(dy) \\ &\equiv I_1 + I_2 + I_3.\end{aligned} \tag{33}$$

By (30) and by the long-tailedness of F, for any fixed A,

$$I_1 \leq \mathbf{P}\{S_{n-1} \in (x-A, x]\} = o(\overline{F}(x)) \quad \text{as } x \to \infty. \tag{34}$$

By (31) and (32) we get, for $A = A(x) \to \infty$,

$$\begin{aligned} I_2 &\leq \int_A^{x+A} \mathbf{P}\{S_{n-1} > x - y\} F(dy) \\ &\leq c \int_A^{x+A} \overline{F}(x-y) F(dy) = o(\overline{F}(x)) \quad \text{as } x \to \infty. \end{aligned} \tag{35}$$

Uniformly in $y \geq x + A(x)$, $\mathbf{P}\{S_{n-1} \in (x-y, x], \tau \geq n\} \to \mathbf{P}\{\tau \geq n\}$ as $x \to \infty$. Thus,

$$I_3 \sim \mathbf{P}\{\tau \geq n\}\overline{F}(x + A(x)) \sim \mathbf{P}\{\tau \geq n\}\overline{F}(x) \quad \text{as } x \to \infty. \tag{36}$$

Substituting (34)–(36) into (33) we get (28).

To prove (29) we note that

$$P_2 \leq \mathbf{P}\{S_{n-1} \in (x, x+A]\} + \mathbf{P}\{S_{n-1} > x + A\} F(-A).$$

As in (34), the first term on the right is of order $o(\overline{F}(x))$. Due to (31), the second term is not greater than $c\overline{F}(x)F(-A)$ where $F(-A)$ can be made as small as we please by the choice of sufficiently large A. The proof is complete.

Here is our general result for a counting random variable with, possibly, unbounded support.

Theorem 10. *Let* $\mathbf{E}|\xi| < \infty$ *and let a counting variable* τ *do not depend on the future. Assume that* $F \in \mathscr{S}^*$ *and that there exists an increasing function* $h(x)$ *such that*

$$\overline{F}(x \pm h(x)) \sim \overline{F}(x) \quad \textit{and} \quad \mathbf{P}\{\tau > h(x)\} = o(\overline{F}(x)) \quad \textit{as } x \to \infty. \tag{37}$$

Then $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ *as* $x \to \infty$.

Proof of Theorem 10 follows from Lemmas 1 and 2 below. Condition (37) is stronger than condition (4). At the end of this section, we provide an example of a stopping time which shows that condition (37) is essential and cannot be weakened to (4).

Lemma 1. *Let* $\mathbf{E}\xi > 0$ *and let a counting variable* τ *do not depend on the future. If* F *is long-tailed then*

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \geq \mathbf{E}\tau.$$

If, in addition, $F \in \mathscr{S}^*$ *and condition* (37) *holds, then* $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ *as* $x \to \infty$.

Proof. Fix a positive integer N and a positive A. The following lower bound holds, for $x > A$:

$$\begin{aligned} \mathbf{P}\{S_\tau > x\} &\geq \sum_{j=1}^N \mathbf{P}\{S_1, \ldots, S_{j-1} \in [-A, A], \xi_j > x + 2A, S_\tau > x, \tau \geq j\} \\ &\geq \sum_{j=1}^N \mathbf{P}\{S_1, \ldots, S_{j-1} \in [-A, A], \xi_j > x + 2A, \min_{i>j}(S_i - S_j) > -A, \tau \geq j\}. \end{aligned}$$

Since $\{\tau \geq j\} = \overline{\{\tau \leq j-1\}}$ and since τ does not depend on the future,

$$\begin{aligned}\mathbf{P}\{S_\tau > x\} &\geq \sum_{j=1}^{N} \mathbf{P}\{S_1,\dots,S_{j-1} \in [-A,A], \tau \geq j\}\mathbf{P}\{\xi_j > x+2A, \min_{i>j}(S_i - S_j) > -A\} \\ &= \overline{F}(x+2A)\mathbf{P}\{\min_{i\geq 1} S_i > -A\}\sum_{j=1}^{N} \mathbf{P}\{S_1,\dots,S_{j-1} \in [-A,A], \tau \geq j\}.\end{aligned}$$

By the long-tailedness of F,

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \geq \mathbf{P}\{\min_{i\geq 1} S_i > -A\}\sum_{j=1}^{N} \mathbf{P}\{S_1,\dots,S_{j-1} \in [-A,A], \tau \geq j\}.$$

Since the mean of ξ is positive, $\mathbf{P}\{\min_{i\geq 1} S_i > -A\} \to 1$ as $A \to \infty$. Hence, for any N,

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \geq \sum_{j=1}^{N} \mathbf{P}\{\tau \geq j\}.$$

Letting now $N \to \infty$ completes the proof of the lower bound.

The upper bound,

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \leq \mathbf{E}\tau,$$

follows from [15, Corollary 3] which states that, under the conditions $F \in \mathscr{S}^*$ and (37), $\mathbf{P}\{M_\tau > x\} \sim \overline{F}(x)\mathbf{E}\tau$ as $x \to \infty$. The proof is complete.

Lemma 2. *Let $\mathbf{E}\xi \leq 0$ and let a counting variable τ do not depend on the future. If $F \in \mathscr{S}^*$, then*

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \leq \mathbf{E}\tau.$$

Under the additional condition (37), $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ *as* $x \to \infty$.

Proof. The upper bound follows from [15, Corollary 3] in the same way as the upper bound in the previous proof. To obtain the lower bound, take any positive ε and consider a random walk $\widetilde{S}_n = S_n + n(|\mathbf{E}\xi| + \varepsilon)$ with a positive drift. We have

$$\begin{aligned}\mathbf{P}\{S_\tau > x\} &= \mathbf{P}\{\widetilde{S}_\tau > x + (|\mathbf{E}\xi|+\varepsilon)\tau\} \\ &\geq \mathbf{P}\{\widetilde{S}_\tau > x + (|\mathbf{E}\xi|+\varepsilon)h(x)\} - \mathbf{P}\{\tau > h(x)\}.\end{aligned}$$

Here the last term in the right side is $o(\overline{F}(x))$ and, by Lemma 1, the first term is equivalent to $\mathbf{E}\tau\overline{F}(x + (|\mathbf{E}\xi|+\varepsilon)h(x)) \sim \mathbf{E}\tau\overline{F}(x)$ as $x \to \infty$. This completes the proof.

For intermediate regularly varying tail distributions, Theorem 10 implies the following

Corollary 3. *Let $\mathbf{E}|\xi| < \infty$ and let a counting variable τ do not depend on the future. Assume that F is an intermediate regularly varying distribution and that*

$$\mathbf{P}\{\tau > x\} = o(\overline{F}(x)) \quad \text{as } x \to \infty. \tag{38}$$

Then $\mathbf{P}\{S_\tau > x\} \sim \mathbf{E}\tau\overline{F}(x)$ *as* $x \to \infty$.

The latter corollary generalises the corresponding result by Greenwood and Monroe [17, Theorem 1] where a regularly varying F and a stopping time τ were considered. In [5, Theorem 2], A. Borovkov and Utev obtained an upper bound for the tail distribution of S_τ assuming that both tail distributions of ξ_1 and of τ are bounded from above by the same dominated varying distribution.

Proof. From condition (38), for any $\varepsilon > 0$,

$$\mathbf{P}\{\tau > \varepsilon x\} \quad = \quad o(\overline{F}(\varepsilon x)) = o(\overline{F}(x)) \quad \text{ as } x \to \infty,$$

since F is intermediate regularly varying. Thus, there exists an increasing function $h(x) = o(x)$ such that $\mathbf{P}\{\tau > h(x)\} = o(\overline{F}(x))$ as $x \to \infty$. Again by the intermediate regular variation of F, for any $h(x) = o(x)$, $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$. So, condition (37) is fulfilled and we can conclude the desired asymptotics from Theorem 10.

We conclude with an example of a stopping time τ showing that condition (37) is essential for the conclusion of Theorem 10. Consider a distribution F on $[1, \infty)$. Take an increasing function $H(x) : \mathbf{R} \to \mathbf{Z}^+$ such that $H(x) < x/2$. The counting random variable $\tau = H(2\xi_1) + 1$ is a stopping time. On the event $\xi_1 > x - H(x)$ we have $\tau \geq H(2(x - H(x))) + 1 \geq H(x) + 1$. Hence,

$$\mathbf{P}\{S_\tau > x\} \geq \mathbf{P}\{\xi_1 > x - H(x), \xi_2 + \ldots + \xi_\tau \geq H(x)\} = \mathbf{P}\{\xi_1 > x - H(x)\},$$

due to $\xi \geq 1$. For a Weibull type distribution, namely $\overline{F}(x) = e^{-x^\beta}$, $0 < \beta < 1$, $x \geq 1$, we can choose $H(x)$ in such a way that $H(x) = o(x)$ and $H(x)/x^{1-\beta} \to \infty$ as $x \to \infty$. Then condition (4) holds, but asymptotics (3) does not, because $\overline{F}(x - H(x))/\overline{F}(x) \to \infty$ and

$$\frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \quad \to \quad \infty.$$

In this example there is no a function $h(x)$ such that condition (37) holds. Indeed, if $\overline{F}(x - h(x)) \sim \overline{F}(x)$ then $h(x) = o(x^{1-\beta})$ and $H^{-1}(h(x) - 1) = o(x)$ which implies

$$\mathbf{P}\{\tau > h(x)\}/\overline{F}(x) \quad = \quad \mathbf{P}\{H(2\xi) > h(x) - 1\}/\overline{F}(x) \to \infty \quad \text{ as } x \to \infty.$$

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