Siberian Mathematical Journal, Vol. 39, No. 3, 1998

MOMENT INEQUALITIES FOR GENERALIZED L-STATISTICS†)

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UDC 517.53

1. Introduction and statement of the main results. Denote by $X_1, \dots, X_n$ independent identically distributed random variables. We study statistics of the type

$$\Phi_n := \sum_{i=1}^{n} h_{ni}(X_{n:i}), \tag{1}$$

where $X_{n:1} \le \dots \le X_{n:n}$ are the order statistics based on the sample $\{X_i;\ i \le n\}$ and $h_{ni} : \mathbf{R} \to \mathbf{R}$, $i = 1, \dots, n$, are measurable functions. In particular, if $h_{ni}(y) = c_{ni}h(y)$ and the function $h(y)$ is monotonic then Φ_n represents the classical L-statistic.

Without loss of generality, we can assume that X_1 has the exponential distribution with parameter 1 because we can define an arbitrary sample $\{Y_i; i \le n\}$ by the formula $Y_i = G^{-1}(F(X_i))$, where F is the distribution function of X_1 and $G^{-1}(z) = \inf\{t : G(t) \ge z\}$ is the quantile transform of the distribution function of Y_1. Since the superposition of the functions G^{-1} and F is monotonic, $G^{-1}(F(X_{n:1})) \le \dots \le G^{-1}(F(X_{n:n}))$ represent the order statistics based on the sample $\{Y_i;\ i \le n\}$. Thus,

$$\sum_{i=1}^{n} h_{ni}(Y_{n:i}) = \sum_{i=1}^{n} \tilde{h}_{ni}(X_{n:i}),$$

where $\tilde{h}_{ni}(x) := h_{ni}(G^{-1}(F(x)))$. Choice of the exponential distribution is explained by the structure of the corresponding order statistics (see Section 2). In particular, in [1] this property provided asymptotic normality of the classical L-statistics.

The functionals (1) of this general form are called *generalized L-statistics.* For the first time, these statistics were introduced in [2, 3]. The Fourier analysis of distributions of Φ_n is contained in [4]. Note that the integral type statistics (integral functionals of the empirical distribution function; for example, the Anderson–Darling–Cramér statistics) can be represented as (1) [2, 4]. The main goal of the present paper is to obtain upper bounds for the moments of Φ_n.

Hereafter, we consider the more convenient form of generalized L-statistics as additive functionals of centered order statistics:

$$\bar{\Phi}_n := \sum_{i=1}^{n} h_{ni}(X_{n:i} - \mathbf{E}X_{n:i}). \tag{2}$$

Theorem 1. *Let the functions $h_{ni}(x), i = 1, \dots, n$, in (2) satisfy the following condition:*

$$|h_{ni}(x)| \le a_{n,i} + b_{n,i}|x|^m \quad \text{for some} \quad m \ge 1, \tag{3}$$

where $a_{n,i}$ and $b_{n,i}$ are positive constants depending only on i and n. Then, for all $r \ge 2$,

$$\mathbf{E}\left|\bar{\Phi}_n\right|^r \le 2^{r-1}A_n^r + 2^{r(1+m)-2}\beta_m^{rm}$$
$$+C(r,m)\left\{\Gamma(rm+1)\sum_{i=1}^{n}\frac{(B_{n,i})^r}{(n+1-i)^{rm}} + \left(\sum_{i=1}^{n}\frac{(B_{n,i})^{2/m}}{(n+1-i)^2}\right)^{rm/2}\right\}, \tag{4}$$

†) The research was financially supported by the Russian Foundation for Basic Research (Grants 96-01-01939 and 96-15-96295) and INTAS (Grant 95-0099).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 39, No. 3, pp. 483–489, May–June, 1998. Original article submitted November 28, 1996.

where $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt$ *is the gamma function,*

$$A_n := \sum_{i=1}^n a_{n,i},\ B_{n,i} := \sum_{j=i}^n b_{n,j},\ \overline{B}_{n,i} := \sum_{j=i}^n b_{n,j}^{2/m},\ \beta_m := \left(\sum_{i=1}^n \frac{\overline{B}_{n,i}}{(n+1-i)^2}\right)^{1/2}$$

if $m \geq 2$, *and*

$$\beta_m := \left(\sum_{i=1}^n \frac{B_{n,i}}{(n+1-i)^2}\right)^{1/2}$$

if $1 \leq m < 2$; $C(r,m) := 2^{r(1+m)-2}(Krm)^{rm}$, *with* K *an absolute positive constant.*

Consider the special case $h_{ni}(x) := x$, $i = 1,\dots,n$. Then $a_{n,i} = 0$, $b_{n,i} = 1$, and $m = 1$. We infer from here that $A_n = 0$, $B_{n,i} = n+1-i$, $\beta_1 = \left(\sum_{k\leq n} 1/k\right)^{1/2}$, and the statistic $\bar{\Phi}_n$ has the form

$$\bar{\Phi}_n = \sum_{i=1}^n (X_i - \mathbf{E}X_i),$$

where $X_1,\dots,X_n$ are independent identically distributed random variables having the exponential law with parameter 1. By the central limit theorem, we have, as $n \to \infty$,

$$\mathbf{E}|\bar{\Phi}_n|^r = n^{r/2}\mathbf{E}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i - \mathbf{E}X_i)\right|^r \sim n^{r/2}\mathbf{E}|\eta|^r = \frac{2^{r/2}}{\sqrt{\pi}}\Gamma\left(\frac{r+1}{2}\right)n^{r/2},$$

where the random variable η has the standard Gaussian distribution. On the other hand, we obtain from (4):

$$\mathbf{E}|\bar{\Phi}_n|^r \leq 2^{2r-2}\beta_1 + 2^{2r-2}K^r r^r \Gamma(r+1)n + 2^{2r-2}K^r r^r n^{r/2}.$$

Taking the factorial rate of increase of the gamma function into account, we conclude that inequality (4) is exact.

Introduce the centered generalized L-statistic

$$\widetilde{\Phi}_n := \sum_{i=1}^n h_{ni}(X_{n:i}) - \sum_{i=1}^n h_{ni}(\mathbf{E}X_{n:i}). \tag{5}$$

The following assertion may be derived from Theorem 1 as an almost immediate corollary.

Theorem 2. *Let the functions* h_{ni}, $i = 1,\dots,n$, *in* (5) *be differentiable and*

$$|h'_{ni}(x)| \leq \alpha_{n,i} + \beta_{n,i}|x|^s \text{ for some } s \geq 1, \tag{6}$$

where $\alpha_{n,i}$ *and* $\beta_{n,i}$ *are some positive constants depending only on* i *and* n. *Then, for all* $r \geq 2$,

$$\mathbf{E}|\widetilde{\Phi}_n|^r \leq 8^{r-1}\left(\sum_{i=1}^n \frac{B_{ni}}{(n+1-i)^2}\right)^{r/2}$$
$$+p_1\left\{\Gamma(r+1)\sum_{i=1}^n \frac{B_{ni}^r}{(n+1-i)^r} + \left(\sum_{i=1}^n \frac{B_{ni}^2}{(n+1-i)^2}\right)^{r/2}\right\}$$
$$+p_2\left(\sum_{i=1}^n \frac{\overline{C}_{ni}}{(n+1-i)^2}\right)^{r(s+1)/2} + p_3\sum_{i=1}^n \frac{C_{ni}^r}{(n+1-i)^{r(s+1)}}. \tag{7}$$

where

$$B_{ni} := \sum_{j=i}^{n}(\alpha_{n,j} + 2^{s-1}\beta_{n,j}\Lambda_j^s), \quad \Lambda_j := \mathbf{E}X_{n:j} = \sum_{k=1}^{j}\frac{1}{n+1-k},$$

$$C_{ni} := \sum_{j=i}^{n}\beta_{n,j}, \quad \widetilde{C}_{ni} := \sum_{j=i}^{n}\beta_{n,j}^{2/(s+1)},$$

$$p_1 := 2^{3(r-1)}(K_1 r)^r, \quad p_2 := 2^{r(2s+1)-3}(1 + (K_2 r(s+1))^{r(s+1)}),$$

$$p_3 := 2^{3(r-1)}(K_2 r(s+1))^{r(s+1)}\Gamma(r(s+1)+1),$$

with K_1 and K_2 some absolute positive constants.

2. Proof of Theorem 1. Define the random variables $\{\tau_{ni}; 1 \le i \le n\}$ by the formula $\tau_{ni} := X_{n:i} - X_{n:i-1}$, $X_{n:0} = 0$. It is known [5] that $\tau_{n1}, \dots, \tau_{nn}$ are independent and have the exponential distributions with the corresponding parameters:

$$P(\tau_{ni} > t) = e^{-(n+1-i)t}, \quad i = 1, \dots, n.$$

Therefore, the order statistics $X_{n:k}$ are represented as the partial sums of the random variables:

$$X_{n:k} = \sum_{i=1}^{k}\tau_{ni}, \quad k = 1, \dots, n.$$

Consider the random process $S_n(t) := \sum_{i \le nt+1}\tau_{ni}$, $0 \le t \le 1$. Put $\widetilde{S}_n(t) := S_n(t) - \mathbf{E}S_n(t)$. Then

$$\widetilde{S}_n(t) = \sum_{i=1}^{n}\xi_i(t), \tag{8}$$

where $\xi_i(t) := (\tau_{ni} - \mathbf{E}\tau_{ni})I\{(i-1)/n \le t\}$. Define the function

$$\varphi_n(t,z) := nh_{ni}(z) \text{ for all } t \in [(i-1)/n, i/n), \quad i = 1, \dots, n. \tag{9}$$

Then the following equality holds:

$$\bar{\Phi}_n = \int_0^1 \varphi_n(t, \widetilde{S}_n(t))\,dt. \tag{10}$$

It follows from (3) that, for all $t \in [(i-1)/n, i/n)$ and $i = 1, \dots, n$,

$$|\varphi_n(t,z)| = n|h_{ni}(z)| \le na_{n,i} + nb_{n,i}|z|^m. \tag{11}$$

Thus, we obtain from (10) and (11)

$$|\bar{\Phi}_n| \le \sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}(na_{n,i} + nb_{n,i}|\widetilde{S}_n(t)|^m)\,dt$$

$$= \sum_{i=1}^{n}a_{n,i} + \sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}nb_{n,i}|\widetilde{S}_n(t)|^m\,dt = A_n + \int_0^1|\widetilde{S}_n(t)|^m\lambda(dt) = A_n + \|\widetilde{S}_n\|^m. \tag{12}$$

where $\lambda(dt) := q(t)\,dt$ with $q(t) := nb_{n,i}$ if $(i-1)/n \le t < i/n$, and $\|\cdot\|$ is the standard norm in $\mathcal{L}_m := \mathcal{L}_m([0,1],\lambda)$. Hence,

$$\begin{aligned}\mathbf{E}\,|\bar{\Phi}_n|^r &\le 2^{r-1}A_n^r + 2^{r-1}\mathbf{E}\|\widetilde{S}_n\|^{rm}\\ &\le 2^{r-1}A_n^r + 2^{r(1+m)-2}\mathbf{E}|\,\|\widetilde{S}_n\| - \mathbf{E}\|\widetilde{S}_n\|\,|^{rm} + 2^{r(1+m)-2}(\mathbf{E}\|\widetilde{S}_n\|)^{rm}.\end{aligned} \tag{13}$$

It is proved in [6] that, for independent centered random variables $Y_1,\dots,Y_n$ in a separable Banach space, the following inequality holds:

$$\mathbf{E}|\,\|S_n\| - \mathbf{E}\|S_n\|\,|^t \le (Kt)^t\left(\sum_{i=1}^n \mathbf{E}\|Y_i\|^t + \left(\sum_{i=1}^n \mathbf{E}\|Y_i\|^2\right)^{t/2}\right), \quad t\ge 2, \tag{14}$$

where K is an absolute positive constant, $S_n := \sum_{i=1}^n Y_i$.

We note that $\xi_1(t),\dots,\xi_n(t)$ are independent nonidentically distributed random variables with mean zero and values in the separable Banach space $\mathcal{L}_m$. Thus, putting $t = rm$ in (14), we obtain

$$\mathbf{E}|\,\|\widetilde{S}_n\| - \mathbf{E}\|\widetilde{S}_n\|\,|^{rm} \le (Krm)^{rm}\left(\sum_{i=1}^n \mathbf{E}\|\xi_i\|^{rm} + \left(\sum_{i=1}^n \mathbf{E}\|\xi_i\|^2\right)^{rm/2}\right). \tag{15}$$

By the definition of the norm in $\mathcal{L}_m$ we have

$$\begin{aligned}\|\xi_i\|^k &= \left(\int_0^1 |\xi_i(t)|^m\lambda(dt)\right)^{k/m} = |\tau_{ni} - \mathbf{E}\tau_{ni}|^k\left(\int_{(i-1)/n}^1 q(t)\,dt\right)^{k/m}\\ &= |\tau_{ni} - \mathbf{E}\tau_{ni}|^k\left(\sum_{k=i}^n b_{n,k}\right)^{k/m} = |\tau_{ni} - \mathbf{E}\tau_{ni}|^k(B_{n,i})^{k/m}, \quad i = 1,\dots,n.\end{aligned}$$

Whence the relation follows:

$$\mathbf{E}\|\xi_i\|^k = (B_{n,i})^{k/m}\mathbf{E}|\tau_{ni} - \mathbf{E}\tau_{ni}|^k, \quad i = 1,\dots,n. \tag{16}$$

Setting $k = 2$ in (16), we obtain

$$\mathbf{E}\|\xi_i\|^2 = (B_{n,i})^{2/m}\mathbf{E}(\tau_{ni} - \mathbf{E}\tau_{ni})^2 = \frac{(B_{n,i})^{2/m}}{(n+1-i)^2}. \tag{17}$$

Now we estimate $\mathbf{E}\|\xi_i\|^k$ for any $k > 2$:

$$\begin{aligned}\mathbf{E}\|\xi_i\|^k &= (B_{n,i})^{k/m}\int_0^\infty \left|x - \frac{1}{n+1-i}\right|^k (n+1-i)e^{-(n+1-i)x}\,dx\\ &= \frac{(B_{n,i})^{k/m}}{e(n+1-i)^k}\int_{-1}^\infty |y|^k e^{-y}\,dy = \frac{(B_{n,i})^{k/m}}{e(n+1-i)^k}\left(\int_0^\infty y^k e^{-y}\,dy + \int_{-1}^0 |y|^k e^{-y}dy\right)\\ &= \frac{(B_{n,i})^{k/m}}{e(n+1-i)^k}\left(\Gamma(k+1) + \int_0^1 z^k e^z\,dz\right) \le \frac{(B_{n,i})^{k/m}}{e(n+1-i)^k}(\Gamma(k+1)+e)\\ &\le \Gamma(k+1)\frac{(B_{n,i})^{k/m}}{(n+1-i)^k}.\end{aligned} \tag{18}$$

Inserting (17) and (18) in (15) we deduce the estimate

$$\mathbf{E}|\,\|\widetilde{S}_n\| - \mathbf{E}\|\widetilde{S}_n\|\,|^{rm}$$
$$\leq (Krm)^{rm}\left(\Gamma(rm+1)\sum_{i=1}^{n}\frac{(B_{n,i})^r}{(n+1-i)^{rm}} + \left(\sum_{i=1}^{n}\frac{(B_{n,i})^{2/m}}{(n+1-i)^2}\right)^{rm/2}\right). \tag{19}$$

Assume that $m \geq 2$. Then

$$\|\widetilde{S}_n\|^2 = \left(\int_0^1\left|\sum_{i=1}^{n}\xi_i(t)\right|^m \lambda(dt)\right)^{2/m} = \left(\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n}\left|\sum_{i=1}^{n}\xi_i(t)\right|^m \lambda(dt)\right)^{2/m}$$
$$= \left(\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n}\left|\sum_{i=1}^{k}(\tau_{ni}-\mathbf{E}\tau_{ni})\right|^m nb_{n,k}\,dt\right)^{2/m} = \left(\sum_{k=1}^{n}b_{n,k}\left|\sum_{i=1}^{k}(\tau_{ni}-\mathbf{E}\tau_{ni})\right|^m\right)^{2/m}$$
$$\leq \sum_{k=1}^{n}b_{n,k}^{2/m}\left(\sum_{i=1}^{k}(\tau_{ni}-\mathbf{E}\tau_{ni})\right)^2.$$

Whence it follows that

$$\mathbf{E}\|\widetilde{S}_n\| \leq (\mathbf{E}\|\widetilde{S}_n\|^2)^{1/2} \leq \left(\sum_{k=1}^{n}b_{n,k}^{2/m}\mathbf{E}\left(\sum_{i=1}^{k}(\tau_{ni}-\mathbf{E}\tau_{ni})\right)^2\right)^{1/2}$$
$$= \left(\sum_{k=1}^{n}b_{n,k}^{2/m}\mathbf{E}\left\{\sum_{i=1}^{k}(\tau_{ni}-\mathbf{E}\tau_{ni})^2 + 2\sum_{1\leq i<j\leq k}(\tau_{ni}-\mathbf{E}\tau_{ni})(\tau_{nj}-\mathbf{E}\tau_{nj})\right\}\right)^{1/2}$$
$$= \left(\sum_{k=1}^{n}b_{n,k}^{2/m}\sum_{i=1}^{k}\frac{1}{(n+1-i)^2}\right)^{1/2} = \left(\sum_{i=1}^{n}\sum_{k=i}^{n}\frac{b_{n,k}^{2/m}}{(n+1-i)^2}\right)^{1/2}$$
$$= \left(\sum_{i=1}^{n}\frac{\widetilde{B}_{n,i}}{(n+1-i)^2}\right)^{1/2} = \beta_m. \tag{20}$$

Now we consider the case $1 \leq m < 2$. Using the Hölder inequality twice, we obtain the following estimate:

$$\mathbf{E}\|\widetilde{S}_n\| = \mathbf{E}\left(\int_0^1\left|\sum_{i=1}^{n}\xi_i(t)\right|^m\lambda(dt)\right)^{1/m} \leq \mathbf{E}\left(\int_0^1\left(\sum_{i=1}^{n}\xi_i(t)\right)^2\lambda(dt)\right)^{1/2}$$
$$= \mathbf{E}\left(\sum_{i=1}^{n}\int_0^1\xi_i^2(t)\lambda(dt) + 2\sum_{1\leq i<j\leq n}\int_0^1\xi_i(t)\xi_j(t)\lambda(dt)\right)^{1/2}$$
$$= \mathbf{E}\left(\sum_{i=1}^{n}B_{n,i}(\tau_{ni}-\mathbf{E}\tau_{ni})^2 + 2\sum_{1\leq i<j\leq n}B_{n,j}(\tau_{ni}-\mathbf{E}\tau_{ni})(\tau_{nj}-\mathbf{E}\tau_{nj})\right)^{1/2}$$

$$\leq \left(\mathbf{E} \left(\sum_{i=1}^{n} B_{n,i} \left(\tau_{ni} - \mathbf{E}\tau_{ni}\right)^2 + 2 \sum_{1\leq i<j\leq n} B_{n,j} \left(\tau_{ni} - \mathbf{E}\tau_{ni}\right) \left(\tau_{nj} - \mathbf{E}\tau_{nj}\right) \right) \right)^{1/2}$$

$$= \left(\sum_{i=1}^{n} \frac{B_{n,i}}{(n+1-i)^2} \right)^{1/2} = \beta_m. \tag{21}$$

Inserting (19)–(21) in (13), we finally derive

$$\mathbf{E} \left|\bar{\Phi}_n\right|^r \leq 2^{r-1} A_n^r + 2^{r(1+m)-2} \beta_m^{rm}$$

$$+2^{r(1+m)-2} (Krm)^{rm} \left\{ \Gamma(rm+1) \sum_{i=1}^{n} \frac{(B_{n,i})^r}{(n+1-i)^{rm}} + \left(\sum_{i=1}^{n} \frac{(B_{n,i})^{2/m}}{(n+1-i)^2} \right)^{rm/2} \right\}.$$

Theorem 1 is proved.

3. Proof of Theorem 2. It follows from the Taylor formula with the integral type remainder that

$$\widetilde{\Phi}_n = \sum_{i=1}^{n} (X_{n:i} - \mathbf{E}X_{n:i}) \int_0^1 h'_{ni}(\mathbf{E}X_{n:i} + \theta(X_{n:i} - \mathbf{E}X_{n:i}))\, d\theta. \tag{22}$$

We have from condition (6)

$$|\widetilde{\Phi}_n| \leq \sum_{i=1}^{n} |X_{n:i} - \mathbf{E}X_{n:i}| \int_0^1 (\alpha_{n,i} + \beta_{n,i} |\mathbf{E}X_{n:i} + \theta(X_{n:i} - \mathbf{E}X_{n:i})|^s)\, d\theta$$

$$\leq \sum_{i=1}^{n} |X_{n:i} - \mathbf{E}X_{n:i}| \int_0^1 \left(\alpha_{n,i} + 2^{s-1} \beta_{n,i} \Lambda_i^s + 2^{s-1} \beta_{n,i} |X_{n:i} - \mathbf{E}X_{n:i}|^s \theta^s\right) d\theta$$

$$= \sum_{i=1}^{n} (\alpha_{n,i} + 2^{s-1} \beta_{n,i} \Lambda_i^s) |X_{n:i} - \mathbf{E}X_{n:i}| + \frac{2^{s-1}}{s+1} \sum_{i=1}^{n} \beta_{n,i} |X_{n:i} - \mathbf{E}X_{n:i}|^{s+1}$$

$$\leq \sum_{i=1}^{n} \left(\alpha_{n,i} + 2^{s-1} \beta_{n,i} \Lambda_i^s\right) |X_{n:i} - \mathbf{E}X_{n:i}| + 2^{s-2} \sum_{i=1}^{n} \beta_{n,i} |X_{n:i} - \mathbf{E}X_{n:i}|^{s+1}.$$

Put

$$R_1 := \sum_{i=1}^{n} (\alpha_{n,i} + 2^{s-1} \beta_{n,i} \Lambda_i^s) |X_{n:i} - \mathbf{E}X_{n:i}|, \quad R_2 := \sum_{i=1}^{n} \beta_{n,i} |X_{n:i} - \mathbf{E}X_{n:i}|^{s+1}.$$

Then

$$\mathbf{E}|\widetilde{\Phi}_n|^r \leq 2^{r-1} \mathbf{E}R_1^r + 2^{r(s-1)-1} \mathbf{E}R_2^r. \tag{23}$$

Using Theorem 1 for R_1 and R_2, we obtain the inequalities

$$\mathbf{E}R_1^r \leq 2^{2r-2} \left(\sum_{i=1}^{n} \frac{B_{ni}}{(n+1-i)^2} \right)^{r/2}$$

$$+2^{2r-2} (K_1 r)^r \left\{ \Gamma(r+1) \sum_{i=1}^{n} \frac{B_{ni}^r}{(n+1-i)^r} + \left(\sum_{i=1}^{n} \frac{B_{ni}^2}{(n+1-i)^2} \right)^{r/2} \right\}. \tag{24}$$

$$\mathbf{E}R_2^r \le 2^{r(s+2)-2}\left(\sum_{i=1}^{n}\frac{\overline{C}_{ni}}{(n+1-i)^2}\right)^{r(s+1)/2}$$

$$+q\left\{\Gamma\Big(r(s+1)+1\Big)\sum_{i=1}^{n}\frac{C_{ni}^r}{(n+1-i)^{r(s+1)}}+\left(\sum_{i=1}^{n}\frac{C_{ni}^{2/(s+1)}}{(n+1-i)^2}\right)^{r(s+1)/2}\right\}, \tag{25}$$

where $q := 2^{r(s+2)-2}(K_2 r(s+1))^{r(s+1)}$. Since $2/(s+1) \le 1$, we have $C_{ni}^{2/(s+1)} \le \overline{C}_{ni}$. Thus,

$$\mathbf{E}R_2^r \le (2^{r(s+2)-2}+q)\left(\sum_{i=1}^{n}\frac{\overline{C}_{ni}}{(n+1-i)^2}\right)^{r(s+1)/2}$$

$$+q\Gamma\Big(r(s+1)+1\Big)\sum_{i=1}^{n}\frac{C_{ni}^r}{(n+1-i)^{r(s+1)}}. \tag{26}$$

Relation (7) follows from (23), (24), and (26). Theorem 2 is proved.

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