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PROBABILITY INEQUALITIES FOR GENERALIZED *L*-STATISTICS

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1. Introduction

Let $X_1, \dots, X_n$ be independent identically distributed random variables. We study statistics of the type

$$\Phi_n = \sum_{i=1}^{n} h_{ni}(X_{n:i}), \tag{1}$$

where $X_{n:1} \le \dots \le X_{n:n}$ are the order statistics based on the sample $\{X_i; i \le n\}$ and $h_{ni} : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$, are measurable functions. In particular, if $h_{ni}(y) = c_{ni} h(y)$ and $h(y)$ is monotone then Φ_n represents the classical L-statistics.

Functionals (1) in this general form are called *generalized L-statistics*. For the first time, these statistics were introduced in [1, 2] where asymptotic expansions for the distributions of these statistics were given in some particular cases. The Fourier analysis of the distributions of Φ_n is contained in [3]. Note that the integral-type statistics (integral functionals of the empirical distribution function, for example, the Anderson–Darling–Cramér statistics) can be represented as (1), but not as the classical L-statistics (see e.g. [1, 3]). The main purpose of this paper is to obtain upper bounds for the tail probability and moments of Φ_n. Exponential bounds for the tail probabilities of the classical L-statistics were obtained in [4] by means of approximation of L-statistics by U-statistics with nondegenerate kernels, which makes it possible to reduce the problem to analogous problems for sums of independent real-valued random variables. The approach of the present paper illustrates the capabilities of multivariate analysis: the problems in question are reduced to analogous problems for sums of independent random elements taking values in a functional Banach space. In the previous paper [5], containing some moment inequalities for generalized L-statistics, we suggested an analogous approach using a special property of the order statistics based on a sample from an exponential distribution. In the present paper, to study generalized L-statistics we essentially use the properties of order statistics based on a sample from the $(0, 1)$-uniform distribution, although we impose no additional restrictions on the sample distribution for the so-called L-statistics with separated kernels which are introduced below.

Note also that the term "generalized L-statistics" was introduced in [6] where a generalization of the classical L-statistics theory was considered in a somewhat different aspect related to another construction of order statistics.

2. Statement of the Main Results

2.1. Additive functionals of centered order statistics. In this section we consider additive functionals of centered and normalized order statistics based on a sample from the $(0, 1)$-uniform distribution:

$$A_n = \sum_{i=1}^{n} h_{ni}(\sqrt{n+1}(X_{n:i} - \mathbf{E}X_{n:i})). \tag{2}$$

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Obviously, every generalized L-statistic can be represented in this form because of unrestricted dependence of the kernels h_{ni} on the subscripts in (1) and the well-known properties of the quantile transforms, although this form merely plays an auxiliary role in our considerations.

Theorem 1. *Let the functions $h_{ni}(x)$, $i = 1, \dots, n$, in (2) satisfy the following condition:*

$$|h_{ni}(x)| \le a_{ni} + b_{ni}|x|^m \quad \text{for some} \quad m \ge 1, \tag{3}$$

where a_{ni} and b_{ni} are positive constants depending only on i and n. Then

$$\mathbf{P}\{A_n \ge y\} \le 4\exp\biggl\{-\frac{(y/2 - \Lambda)^{2/m} - 2\beta y^{1/m}}{2(B^2 + Hy^{1/m})}\biggr\}, \tag{4}$$

where

$$\beta = C(m)(n+1)^{-1/2}\Biggl(\sum_{i=1}^{n} i^{m/2} b_{ni}\Biggr)^{1/m},$$

$$C(m) = \begin{cases} 1, & \text{if} \quad 1 \le m < 2, \\ (1 + \Gamma(m+1))^{\frac{1}{m}} \max\{1 + \frac{m}{2}; (2e)^{\frac{1}{m}}((1 + \frac{m}{2})e)^{\frac{1}{2}}\}, & \text{if} \quad m \ge 2, \end{cases}$$

$\Gamma(x)$ is the gamma-function, $\Lambda = \sum_{i=1}^{n} a_{ni}$, and

$$B^2 = 2(n+1)^{-1}\sum_{i=1}^{n}\Biggl(\sum_{j=i}^{n} b_{nj}\Biggr)^{2/m}, \quad H = (n+1)^{-1/2}\Biggl(\sum_{i=1}^{n} b_{ni}\Biggr)^{1/m}.$$

Consider the special case $h_{ni}(x) = |x|^m/(n+1)$, $m \ge 2$. Then $a_{ni} = 0$, $b_{ni} = (n+1)^{-1}$, and the statistic A_n has the form

$$A_n = \int_0^1 |G_n(t)|^m \, dt,$$

where $G_n(t)$ is the quantile empirical process based on a sample from the uniform distribution on $[0, 1]$ (see Section 3 for more detail). It is well known (see, for example, [7]) that, as $n \to \infty$, the distributions of the processes $G_n(t)$ converge weakly in the space $D[0, 1]$ to the distribution of a "Brownian bridge" $w^0(t)$. Thus, we have

$$\mathbf{P}\{A_n \ge y\} = \mathbf{P}\biggl\{\int_0^1 |G_n(t)|^m dt \ge y\biggr\} \to \mathbf{P}\{\|w^0\| \ge y^{1/m}\} \quad \text{as } n \to \infty,$$

where $\|\cdot\|$ is the standard norm in $\mathscr{L}_m([0, 1], dt)$. From an inequality in [8] for Gaussian random elements of an arbitrary Banach space, we can obtain the following unimprovable estimate:

$$\mathbf{P}\{\|w^0\| \ge y^{1/m}\} \le \exp\biggl\{-\frac{(y^{1/m} - \sigma)^2}{2\sigma^2}\biggr\},$$

where $y \ge \sigma^m$, $\sigma^m = 2^{m/2}\pi^{-1/2}\Gamma((m+1)/2)B(m/2+1, m/2+1)$, and $B(x, y)$ is the beta-function.

On the other hand, from (4) we obtain the upper bound

$$\mathbf{P}\{A_n \ge y\} \le 4\exp\biggl\{-\frac{y^{2/m} - 4C(m)y^{1/m}}{4(1 + y^{1/m}n^{-1/2})}\biggr\}.$$

So, in this case inequality (4) is exact in some sense.

Consider the classical L-statistic with the kernel $h_{ni}(x) = x(n(n+1))^{-1/2}$. In this case the statistic A_n has the following form:

$$A_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mathbf{E}X_i),$$

where $X_1, \dots, X_n$ are independent random variables distributed uniformly on $[0,1]$. Using (4) with $a_{ni} = 0$, $b_{ni} = (n(n+1))^{-1/2}$, and $m = 1$, we obtain the upper bound

$$\mathbf{P}\{A_n \geq y\} \leq 4 \exp\left\{-\frac{y^2 - 16y/3}{8(2/3 + yn^{-1/2})}\right\},$$

which is rather close to the right-hand side of the classical Bernstein inequality for sums of independent bounded random variables. We compare this result with the following estimate for the tail probability of the classical L-statistic in [4]:

$$\mathbf{P}\{A_n \geq y\} \leq \exp\left\{-\frac{C_0 y^2}{1 + y^{3/2} n^{-1/4}}\right\},$$

where the absolute constant C_0 can be calculated explicitly. It is easy to see that the logarithmic asymptotics of the right-hand side of the last inequality coincides in the order of magnitude with the analogous asymptotics of the right-hand side of the Bernstein inequality only in the range $y = O(n^{1/6})$.

Introduce the centered generalized L-statistic

$$\overline{\Phi}_n = \sum_{i=1}^{n} h_{ni}(X_{n:i}) - \sum_{i=1}^{n} h_{ni}(\mathbf{E}X_{n:i}). \tag{5}$$

The following is immediate from Theorem 1.

Corollary 1. *Let the functions h_{ni}, $i = 1, \dots, n$, in* (5) *satisfy the Lipschitz condition with the respective constants b_{ni}. Then*

$$\mathbf{P}\{\overline{\Phi}_n \geq y\} \leq 4 \exp\left\{-\frac{y^2 - 8\beta_1 y}{8(B_1^2 + H_1 y)}\right\}, \tag{6}$$

where

$$\beta_1 = \frac{1}{n+1} \sum_{i=1}^{n} i^{1/2} b_{ni}, \quad B_1^2 = \frac{2}{(n+1)^2} \sum_{i=1}^{n} \left(\sum_{j=i}^{n} b_{nj} \right)^2, \quad H_1 = \frac{1}{n+1} \sum_{i=1}^{n} b_{ni}.$$

Inequality (6) follows from the relation

$$|\overline{\Phi}_n| \leq \sum_{i=1}^{n} b_{ni} |X_{n:i} - \mathbf{E}X_{n:i}| = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n} b_{ni} |\sqrt{n+1}(X_{n:i} - \mathbf{E}X_{n:i})| \tag{7}$$

and Theorem 1.

The next two assertions contain moment inequalities for the above statistics.

Theorem 2. *Under the conditions of Theorem* 1, *for all $r \geq 2$*

$$\mathbf{E}|A_n|^r \leq 4^r \left\{ \left(\sum_{i=1}^{n} a_{ni} \right)^r + 2^{rm-1} \beta^{rm} \right\} + \frac{2^{r(m+2)-1}}{(n+1)^{rm/2}} (Krm)^{rm} \{\Gamma(rm+1) B_{n,r} + B_{n,2/m}^{rm/2}\}, \tag{8}$$

where

$$B_{n,r} = \sum_{i=1}^{n} \left(\sum_{j=i}^{n} b_{nj} \right)^r,$$

K is an absolute positive constant, and β is defined in Theorem 1.

Corollary 2. *Under the conditions of Corollary 1, for all $r \geq 2$*

$$\mathbf{E}|\overline{\Phi}_n|^r \leq \frac{2^{3r-1}}{(n+1)^r}\left\{\left(\sum_{i=1}^n i^{1/2} b_{ni}\right)^r + (K_1 r)^r \big(\Gamma(r+1) B_{n,r} + B_{n,2}^{r/2}\big)\right\}, \tag{9}$$

where K_1 is an absolute positive constant.

Relation (9) follows from (7) and (8).

We again consider the special case $h_{ni}(x) = x(n(n+1))^{-1/2}$ in which

$$A_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - \mathbf{E}X_i).$$

It is shown in [9] that, for independent random variables $\zeta_1, \dots, \zeta_n$ with mean zero and for all $c > r/2$, the following inequality holds:

$$\mathbf{E}\left|\sum_{i=1}^n \zeta_i\right|^r \leq c^r \sum_{i=1}^n \mathbf{E}|\zeta_i|^r + r c^{r/2} e^c B(r/2, c - r/2) \left(\sum_{i=1}^n \mathbf{E}\zeta_i^2\right)^{r/2}. \tag{10}$$

Putting in (10) $\zeta_i = X_i - \mathbf{E}X_i$, $c = 1 + r/2$, and noting that $\mathbf{E}(X_1 - \mathbf{E}X_1)^2 = 1$, $\mathbf{E}|X_1 - \mathbf{E}X_1|^r \leq \Gamma(r+1)$, we obtain

$$\mathbf{E}|A_n|^r \leq 2(1 + r/2)^{r/2} e^{1+r/2} + (1 + r/2)^r \Gamma(r+1) n^{1-r/2}.$$

On the other hand, from (8) we deduce the upper bound

$$\mathbf{E}|A_n|^r \leq 2^{3r-1}(1 + (Kr)^r) + 2^{3r-1}(Kr)^r \Gamma(r+1) n^{1-r/2},$$

and this estimate is close to that above.

2.2. *L*-statistics with separated kernels. We now consider the following L-statistics with separated kernels:

$$L_n = \sum_{i=1}^n c_{ni} h(X_{n:i}), \tag{11}$$

where c_{ni}, $i = 1, \dots, n$, are some constants, h is an arbitrary measurable (not necessarily monotone) function, and X_1 has an arbitrary distribution function F.

Without loss of generality, we assume that $\sum_{i=1}^n c_{ni} = 0$, since the statistic L_n can be represented in the following form:

$$L_n = \sum_{i=1}^n \tilde{c}_{ni} h(X_{n:i}) + \tilde{c}_n \sum_{i=1}^n h(X_i), \tag{12}$$

where $\tilde{c}_{ni} = c_{ni} - \tilde{c}_n$, $\tilde{c}_n = n^{-1} \sum_{i=1}^n c_{ni}$, and the second term on the right-hand side of (12) is a sum of independent identically distributed random variables for which the moment inequalities and estimates of the tail probability are well known.

It is worth noting that L-statistics of the form (11) were studied by many authors under various restrictions on the weights c_{ni} and the distribution function F. Asymptotic normality of these statistics was investigated mainly in the case of a monotone $h(x)$ (or, in an equivalent setting, $h(x) = x$ and the sample distribution is arbitrary; see, for example, [10–14]). Note that the authors of [13] also used multivariate arguments for asymptotic analysis of such L-statistics. The behavior of large and moderate deviations of L_n was studied in [4, 15, 16].

As noted in [3], statistics of the form (11) with smooth kernels can be represented as integral-type functionals of the empirical distribution function $F_n(t)$. Indeed, suppose that $h \in C^1(\mathbb{R})$ and denote by $\phi_n(x)$ an arbitrary continuous function on $[0,1]$ satisfying the following conditions:

$$\phi_n(0) = 0, \quad \phi_n(k/n) = \sum_{i=1}^{k} c_{ni}, \quad k = 1, \dots, n.$$

The condition $\sum_{i=1}^{n} c_{ni} = 0$ implies the equality $\phi_n(1) = 0$. Integrating by parts, we then obtain

$$L_n = \sum_{i=1}^{n} \left\{ \phi_n\left(\frac{i}{n}\right) - \phi_n\left(\frac{i-1}{n}\right) \right\} h(X_{n:i}) = \int_{\mathbb{R}} h(t)\, d\phi_n(F_n(t)) = -\int_{\mathbb{R}} \phi_n(F_n(t)) h'(t)\, dt.$$

Similar representations can be actually found in many papers dealing with asymptotic analysis of L-statistics.

Define the function $\phi_n(x)$ as follows:

$$\phi_n(x) = nc_{nk}x + \sum_{i=1}^{k} c_{ni} - kc_{nk}, \quad \frac{k-1}{n} < x < \frac{k}{n}, \quad k = 1, \dots, n.$$

Obviously, the function $\phi_n(x)$ satisfies the Lipschitz condition with the constant nc_n, where $c_n = \max_{1 \le k \le n} |c_{nk}|$. Put

$$\gamma_n = \int_{\mathbb{R}} \phi_n(F(t)) h'(t)\, dt.$$

Then we have

$$L_n + \gamma_n = \int_{\mathbb{R}} \{\phi_n(F(t)) - \phi_n(F_n(t)\} h'(t)\, dt. \tag{13}$$

We will use the following notation:

$$g(t,z) = \begin{cases} F(t) & \text{if} \quad t \le z, \\ 1 - F(t) & \text{if} \quad t > z, \end{cases}$$

$$\alpha_k \equiv \alpha(k, F, h) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(t,z) |h'(t)|\, dt \right)^k dF(z),$$

$$H_F = \int_{\mathbb{R}} (F(t)(1 - F(t)))^{1/2} |h'(t)|\, dt.$$

The conditions of Theorems 3 and 4 (see below) contain moment restrictions in terms of H_F and α_k in particular. Thus, the properties of these characteristics as well as the comparison of the above-mentioned restrictions with the classical moment conditions of summation theory are of special interest.

Proposition. *The following hold:*
1. $\mathbf{E} g^2(t, X_1) = F(t)(1 - F(t)$.
2. $\alpha_1 \le H_F$.
3. $\alpha_k \ge \alpha_1^k$, $k \ge 1$.
4. *If h is a monotone function then* $\alpha_k \le 2^k \mathbf{E} |h(X_1)|^k$, $k \ge 1$.
5. *Assume that* $\delta_1 \le |h'(t)| \le \delta_2$ *for some positive* δ_1 *and* δ_2 *and that* $H_F < \infty$. *Then*

$$\mathbf{P}(|X_1| \ge x) = o(x^{-2}) \quad \text{as } x \to \infty.$$

In particular, $\mathbf{E}|X_1|^2(\log^+|X_1|)^{-1-\varepsilon} < \infty$ *for all* $\varepsilon > 0$.

6. *Assume that* $|h'(t)| \le \delta|t|^\beta$ *for some* $\delta > 0$ *and* $\beta \ge 0$ *and that*

$$\mathbf{E}|X_1|^{2(1+\beta)}(\log^+|X_1|)^{2+\varepsilon} < \infty$$

for some $\varepsilon > 0$. *Then* $H_F < \infty$.

REMARK. As follows from the Proposition, existence of H_F and existence of the second moment of the sample are close conditions. Let us study this relation in more detail. Suppose that the function h satisfies the following condition: $\delta_1 \le |h'(t)| \le \delta_2$ for some positive δ_1 and δ_2. In this case, finiteness of H_F is equivalent to that of $\widetilde{H}_F = \int_{\mathbb{R}} \sqrt{F(t)(1-F(t))}dt$.

It is noted in [14, p. 686] that, if the distribution function F has regularly varying tails, then existence of the finite second moment and finiteness of $\widetilde{H}_F$ are equivalent conditions. However, this statement is false. For simplicity, we consider the case in which the left and right tails of the distribution function F behave at the infinities as $|t|^{-p}L(|t|)$, $p > 0$, where $L(t)$ is a slowly varying function. Then existence of the finite second moment and that of $\widetilde{H}_F$ amount to convergence, for some $t_0 > 0$, of the respective integrals

$$\int_{t_0}^{\infty} t^{1-p}L(t)\,dt, \qquad \int_{t_0}^{\infty} t^{-p/2}L^{1/2}(t)\,dt.$$

These integrals converge simultaneously for $p > 2$ and diverge for $p < 2$ (see, for example, [17]). Consider the case $p = 2$. In this case, existence of the second moment follows from finiteness of $\widetilde{H}_F$. Indeed, if $f(x)$ is a nonnegative decreasing function and the integral $\int_0^\infty f(x)dx$ converges, then $f(x) = o(1/x)$ as $x \to \infty$. This follows from the relation

$$0 \le xf(x) \le 2\int_{x/2}^{x} f(t)\,dt \to 0, \quad x \to \infty.$$

Therefore, if

$$\int_{t_0}^{\infty} t^{-1}L^{1/2}(t)\,dt < \infty,$$

then $L(x) \to 0$ as $x \to \infty$; i.e., $L(x) < 1$ for sufficiently large x. It follows that $L(x) \le L^{1/2}(x)$ for sufficiently large x and, consequently,

$$\int_{t_0}^{\infty} t^{-1}L(t)\,dt < \infty.$$

The converse statement is false. Indeed, let $L(t) = C\log^{-q} t$, $1 < q < 2$. Then the second moment obviously exists while $\widetilde{H}_F = \infty$. In other words, in the class of the distribution functions having regularly varying tails, finiteness of $\widetilde{H}_F$ is a stronger condition than existence of the finite second moment.

Theorem 3. *Let the function* $h(x)$ *in* (11) *be continuously differentiable and* $H_F < \infty$. *If* $\alpha_k < \infty$ *for some* $k \ge 1$ *then for all* $y > y_0$

$$\mathbf{P}\{L_n + \gamma_n \ge y\} \le \exp\left\{-\frac{\log 3}{2}\left(\frac{y-y_0}{2y_0}\right)^{\frac{\log 2}{\log 3}}\right\} + \frac{6^{k+2}c_n^k n\alpha_k}{2(y-y_0)^k}, \tag{14}$$

where $y_0 = 24H_F c_n\sqrt{n}$ *if* $\alpha_k \le (24H_F)^k n^{k/2-1}/36$, *and* $y_0 = c_n(36n\alpha_k)^{1/k}$ *if* $\alpha_k > (24H_F)^k n^{k/2-1}/36$.

If $\alpha_k \le k!B^2H^{k-2}/2$ *for some constants* B^2 *and* $H > 0$ *and for every integer* $k \ge 2$ *then*

$$\mathbf{P}\{L_n + \gamma_n \ge y\} \le \exp\left\{-\frac{y^2 - 2H_F c_n\sqrt{n}y}{2c_n(nc_nB^2 + yH)}\right\}. \tag{15}$$

If $|X_1| \le b$ *almost surely then*

$$\mathbf{P}\{L_n + \gamma_n \ge y\} \le \exp\left\{-\frac{(y - H_F c_n\sqrt{n})^2}{2nc_n^2H_0^2}\right\}, \tag{16}$$

where $H_0 = \int_{-b}^{b} |h'(t)|dt$.

Corollary 3. *Let the function* $h(x)$ *in* (11) *be monotone and continuously differentiable,* $H_F < \infty$*, and* $\mathbf{E}|h(X_1)|^k \le k!B^2H^{k-2}/2$ *for every integer* $k \ge 2$ *and some positive constants* B *and* H*. Then*

$$\mathbf{P}\{L_n + \gamma_n \ge y\} \le \exp\left\{-\frac{y^2 - 2H_F c_n\sqrt{n}y}{4c_n(2nc_nB^2 + yH)}\right\}. \tag{17}$$

Relation (17) follows from (15) and the inequality $\alpha_k \le 2^k\mathbf{E}|h(X_1)|^k$ (see the Proposition).

Theorem 4. *Let the function* $h(x)$ *in* (11) *be continuously differentiable,* $H_F < \infty$*, and* $\alpha_k < \infty$ *for some* $k \ge 2$*. Then*

$$\mathbf{E}|L_n + \gamma_n|^k \le 2^{k-1}c_n^k\big((Ck)^k\alpha_k + H_F^k\big)n^{k/2}, \tag{18}$$

where C *is an absolute positive constant.*

3. Proofs of the Main Results

3.1. Proofs of Theorems 1 and 2.

Proof of Theorem 1. Define the random process $G_n(t)$ and the function $\varphi_n(t, z)$ as follows: For all $t \in [i/(n+1), (i+1)/(n+1))$, $i = 0, 1, \dots, n$, we put

$$G_n(t) = \sqrt{n+1}(X_{n:i} - \mathbf{E}X_{n:i}), \quad \varphi_n(t, z) = (n+1)h_{ni}(z), \quad X_{n:0} \equiv 0,\ h_{n0} \equiv 0.$$

Then the following equality holds:

$$A_n = \int_0^1 \varphi_n(t, G_n(t))\,dt. \tag{19}$$

Let $\nu_1, \dots, \nu_{n+1}$ be independent random variables having the exponential law with parameter 1. Put $\tau_i = \nu_i - 1$. Obviously, $\mathbf{E}\tau_i = 0$, $\mathbf{E}\tau_i^2 = 1$. We construct the partial sum process $S_{n+1}(t)$ using the random variables $\{\tau_i\}_{i=1}^{n+1}$:

$$S_{n+1}(t) = \frac{S_k}{\sqrt{n+1}}, \quad \text{if} \quad \frac{k}{n+1} \le t < \frac{k+1}{n+1},$$
$$k = 0, 1, \dots, n, \quad S_{n+1}(1) = \frac{S_{n+1}}{\sqrt{n+1}},$$

where $S_k = \sum_{i=1}^k \tau_i$, $S_0 = 0$. We also consider the conditional partial sum process $S^0_{n+1}(t)$ with the right endpoint fixed at 0. In other words, $S^0_{n+1}(t)$ is a random process with finite-dimensional distributions coinciding with those of the process $S_{n+1}(t)$ under the condition $S_{n+1}(1) = 0$, i.e., for all $0 < t_1 < t_2 < \dots < t_k < 1$,

$$\mathbf{P}(S^0_{n+1}(t_1) < x_1, \dots, S^0_{n+1}(t_k) < x_k) = \mathbf{P}(S_{n+1}(t_1) < x_1, \dots, S_{n+1}(t_k) < x_k | S_{n+1}(1) = 0).$$

The following assertion is proven in [7].

Lemma 1. *The vectors* $\{G_n(\frac{i}{n+1})\}_{i=1}^n$ *and* $\{S_{n+1}^0(\frac{i}{n+1})\}_{i=1}^n$ *coincide in distribution.*

Thus, we have

$$\mathbf{P}\{A_n \geq y\} = \mathbf{P}\biggl\{\int_0^1 \varphi_n(t, G_n(t))\,dt \geq y\biggr\} = \mathbf{P}\biggl\{\int_0^1 \varphi_n\bigl(t, S_{n+1}^0(t)\bigr)\,dt \geq y\biggr\}$$

$$\leq \mathbf{P}\biggl\{\int_0^{\frac{N+1}{n+1}} \varphi_n\bigl(t, S_{n+1}^0(t)\bigr)\,dt \geq \frac{y}{2}\biggr\} + \mathbf{P}\biggl\{\int_{\frac{N+1}{n+1}}^1 \varphi_n\bigl(t, S_{n+1}^0(t)\bigr)\,dt \geq \frac{y}{2}\biggr\}$$

$$= \mathbf{P}\biggl\{\int_0^{\frac{N+1}{n+1}} \varphi_n\bigl(t, S_{n+1}^0(t)\bigr)\,dt \geq \frac{y}{2}\biggr\} + \mathbf{P}\biggl\{\int_{\frac{N+1}{n+1}}^1 \varphi_n\bigl(t, -\bigl(S_{n+1}^0(1) - S_{n+1}^0(t)\bigr)\bigr)\,dt \geq \frac{y}{2}\biggr\}, \tag{20}$$

where N is the integral part of $n/2$. Put

$$P_1 = \mathbf{P}\biggl\{\int_0^{\frac{N+1}{n+1}} \varphi_n\bigl(t, S_{n+1}^0(t)\bigr)\,dt \geq \frac{y}{2}\biggr\},$$

$$P_2 = \mathbf{P}\biggl\{\int_{\frac{N+1}{n+1}}^1 \varphi_n\bigl(t, -\bigl(S_{n+1}^0(1) - S_{n+1}^0(t)\bigr)\bigr)\,dt \geq \frac{y}{2}\biggr\}.$$

Lemma 2. *For all* $n \geq 5$,

$$P_1 \leq 2\mathbf{P}\biggl\{\int_0^{\frac{N+1}{n+1}} \varphi_n\Bigl(t, S_{n+1}(t)\Bigr)\,dt \geq \frac{y}{2}\biggr\},$$

$$P_2 \leq \sqrt{3}\mathbf{P}\biggl\{\int_{\frac{N+1}{n+1}}^1 \varphi_n(t, -\Bigl(S_{n+1}(1) - S_{n+1}(t)))\,dt \geq \frac{y}{2}\biggr\}.$$

PROOF. It was shown in [7] that, for each event $\mathscr{F}$ in the σ-algebra generated by paths of the process $S_{n+1}^0(t)$ until the time moment $1-v$, the following inequality holds:

$$\mathbf{P}\bigl(S_{n+1}^0(\cdot) \in \mathscr{F}\bigr) \leq C\mathbf{P}(S_{n+1}(\cdot) \in \mathscr{F}),$$

where $C = \sup f_1(-x)/f_2(0)$, f_1 and f_2 are probability densities of the random variables $S_{n+1}(1) - S_{n+1}(1-v)$ and $S_{n+1}(1)$ respectively. Since

$$f_1(-x) = \sqrt{n+1}(l - x\sqrt{n+1})^{l-1}\exp\{x\sqrt{n+1} - l)\}/(l-1)!,$$

$$f_2(0) = (n+1)^{n+3/2}e^{-(n+1)}/(n+1)!,$$

where $l = v(n+1)$, we have

$$C = \frac{(n+1)!e^{n+1}(v(n+1)-1)^{(v(n+1)-1)}}{(n+1)^{n+1}(v(n+1)-1)!e^{v(n+1)-1}},$$

because the function $x^N e^{-x}$ takes a maximal value at the point $x = N$. Using the Stirling formula $k! = (2\pi k)^{1/2}(k/e)^k e^{\theta(k)}$, where $1/(12k+1) < \theta(k) < 1/(12k)$ (see, for example, [18]), we finally obtain

$$C \le \left(v - \frac{1}{n+1}\right)^{-1/2}.$$

Taking the obvious symmetry into account, we can use the analogous arguments for evaluating P_2. Substituting $(n-N)/(n+1)$ and $(N+1)/(n+1)$ for v, we obtain the corresponding inequalities. Lemma 2 is proven.

From (20) and Lemma 2 we now derive the estimate

$$\mathbf{P}\{A_n \ge y\} \le 2\mathbf{P}\left\{ \int_0^{\frac{N+1}{n+1}} \varphi_n(t, S_{n+1}(t))\, dt \ge \frac{y}{2} \right\}$$
$$+\sqrt{3}\mathbf{P}\left\{ \int_{\frac{N+1}{n+1}}^{1} \varphi_n(t, -(S_{n+1}(1) - S_{n+1}(t)))\, dt \ge \frac{y}{2} \right\}. \tag{21}$$

We evaluate each summand on the right-hand side of (21). From (3) it follows that $|\varphi_n(t, z)| \le (n+1)(a_{ni} + b_{ni}|z|^m)$ for all $t \in [i/(n+1), (i+1)/(n+1))$, $i = 1, \ldots, n$. Then

$$\int_0^{\frac{N+1}{n+1}} \varphi_n(t, S_{n+1}(t))\, dt \le \sum_{i=1}^{N} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} |\varphi_n(t, S_{n+1}(t))|\, dt \le \sum_{i=1}^{N} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} (n+1)\{a_{ni} + b_{ni}|S_{n+1}(t)|^m\}\, dt$$
$$= \sum_{i=1}^{N} a_{ni} + \int_0^1 |S_{n+1}(t)|^m \lambda(dt) = \sum_{i=1}^{N} a_{ni} + \|S_{n+1}(t)\|_\lambda^m, \tag{22}$$

where $\|\cdot\|_\lambda$ is the standard norm in $\mathscr{L}_m([0,1], \lambda)$, $\lambda(dt) = q_1(t)dt$, $q_1(t) = (n+1)b_{ni}$ if $t \in [i/(n+1), (i+1)/(n+1))$, $i = 1, \ldots, N$, and $q_1(t) = 0$ for other t.

By analogy with the above, we have

$$\int_{\frac{N+1}{n+1}}^{1} \varphi_n(t, -(S_{n+1}(1) - S_{n+1}(t)))\, dt \le \sum_{i=N+1}^{n} a_{ni} + \|\widetilde{S}_{n+1}(t)\|_\mu^m, \tag{23}$$

where $\tilde{S}_{n+1}(t) = S_{n+1}(1) - S_{n+1}(t)$, $\|\cdot\|_\mu$ is the standard norm in $\mathscr{L}_m([0,1], \mu)$, $\mu(dt) = q_2(t)dt$, $q_2(t) = (n+1)b_{ni}$ if $t \in [i/(n+1), (i+1)/(n+1))$, $i = N+1, \ldots, n$, and $q_2(t) = 0$ if $t \in [0, (N+1)/(n+1))$. Substituting (22) and (23) into (21), we obtain

$$\mathbf{P}\{A_n \ge y\} \le 2\mathbf{P}\left\{ \|S_{n+1}(t)\|_\lambda^m \ge \frac{y}{2} - \sum_{i=1}^{N} a_{ni} \right\} + \sqrt{3}\mathbf{P}\left\{ \|\widetilde{S}_{n+1}(t)\|_\mu^m \ge \frac{y}{2} - \sum_{i=N+1}^{n} a_{ni} \right\}.$$

It was proven in [8] that, if independent random variables $Y_1, \ldots, Y_n$ in a separable Banach space satisfy

$$\sum_{j=1}^{n} \mathbf{E}\|Y_j\|^k \le k! B^2 H^{k-2}/2, \quad k = 2, 3, \ldots, \tag{24}$$

for some constants B and $H > 0$, then the following estimate holds:

$$\mathbf{P}(\|Y_1 + \cdots + Y_n\| - \beta \geq x) \leq \exp\biggl\{-\frac{x^2}{2(B^2 + xH)}\biggr\},$$

where $\beta = \mathbf{E}\|Y_1 + \cdots + Y_n\|$. It follows that

$$\mathbf{P}(\|Y_1 + \cdots + Y_n\| \geq x) \leq \exp\biggl\{-\frac{x^2 - 2\beta x}{2(B^2 + xH)}\biggr\}. \tag{25}$$

We note that the random processes $S_{n+1}(t)$ and $\tilde{S}_{n+1}(t)$ can be represented as sums of independent nonidentically distributed random variables with mean zero and values in the corresponding separable Banach spaces $\mathscr{L}_m(\cdot)$:

$$S_{n+1}(t) = \sum_{i=1}^{n+1} \xi_i(t) \quad \text{and} \quad \widetilde{S}_{n+1}(t) = \sum_{i=1}^{n+1} \eta_i(t),$$

where

$$\xi_i(t) = \frac{\tau_i}{\sqrt{n+1}}\mathbf{I}\biggl\{\frac{i}{n+1} \leq t\biggr\}, \quad \eta_i(t) = \frac{\tau_i}{\sqrt{n+1}}\mathbf{I}\biggl\{\frac{i}{n+1} > t\biggr\}.$$

We also note that

$$\|S_{n+1}(t)\|_\lambda = \Biggl\|\sum_{i=1}^{N} \xi_i(t)\Biggr\|_\lambda, \quad \|\widetilde{S}_{n+1}(t)\|_\mu = \Biggl\|\sum_{i=N+2}^{n+1} \eta_i(t)\Biggr\|_\mu.$$

Finally, we have to verify that the random variables $\xi_1(t), \ldots, \xi_{n+1}(t)$ and $\eta_1(t), \ldots, \eta_{n+1}(t)$ satisfy (24). By the definition of the norm in $\mathscr{L}_m([0,1], \lambda)$, we have

$$\|\xi_i(t)\|_\lambda^m = \int_0^1 |\xi_i(t)|^m \, \lambda(dt) = \int_{\frac{i}{n+1}}^{\frac{N+1}{n+1}} \frac{|\tau_i|^m}{(n+1)^{m/2}} q_1(t)\, dt$$

$$= \sum_{j=i}^{N} \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} \frac{|\tau_i|^m}{(n+1)^{m/2}}(n+1) b_{nj}\, dt = \frac{|\tau_i|^m}{(n+1)^{m/2}} \sum_{j=i}^{N} b_{nj}, \quad i = 1, \ldots, N.$$

Whence it follows that

$$\mathbf{E}\|\xi_i(t)\|_\lambda^k \leq k! \frac{\biggl(\sum\limits_{j=i}^{N} b_{nj}\biggr)^{k/m}}{(n+1)^{k/2}}, \quad k = 2, 3, \ldots, \quad i = 1, \ldots, N.$$

It is easy to verify that

$$\sum_{i=1}^{N} \mathbf{E}\|\xi_i(t)\|_\lambda^k \leq k! B^2 H^{k-2}/2, \quad k = 2, 3, \ldots,$$

where

$$B^2 = \frac{2}{n+1} \sum_{i=1}^{n} \Biggl(\sum_{j=i}^{n} b_{nj}\Biggr)^{2/m}, \quad H = \frac{1}{\sqrt{n+1}} \Biggl(\sum_{i=1}^{n} b_{ni}\Biggr)^{1/m}.$$

We now estimate $\mathbf{E}\|S_{n+1}(t)\|_\lambda$. Let $m \geq 2$. Then

$$\|S_{n+1}(t)\|_\lambda^2 = \Bigg(\int_{\frac{1}{n+1}}^{\frac{N+1}{n+1}} \Big| \sum_{j=1}^{N} \xi_j(t) \Big|^m \lambda(dt) \Bigg)^{2/m} = \Bigg(\sum_{i=1}^{N} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \Big| \sum_{j=1}^{i} \frac{\tau_j}{\sqrt{n+1}} \Big|^m (n+1) b_{ni}\, dt \Bigg)^{2/m}$$

$$= \Bigg(\sum_{i=1}^{N} b_{ni} \Big| \sum_{j=1}^{i} \frac{\tau_j}{\sqrt{n+1}} \Big|^m \Bigg)^{2/m} = \frac{1}{n+1} \Bigg(\sum_{i=1}^{N} b_{ni} \Big| \sum_{j=1}^{i} \tau_j \Big|^m \Bigg)^{2/m}.$$

It follows that

$$\mathbf{E}\|S_{n+1}(t)\|_\lambda \leq \big(\mathbf{E}\|S_{n+1}(t)\|_\lambda^2\big)^{1/2} \leq \frac{1}{\sqrt{n+1}} \Bigg(\sum_{i=1}^{N} b_{ni} \mathbf{E} \Big| \sum_{j=1}^{i} \tau_j \Big|^m \Bigg)^{1/m}. \tag{26}$$

Putting in (10) $c = 1 + m/2$, we obtain

$$\mathbf{E} \Big| \sum_{j=1}^{i} \tau_j \Big|^m \leq C_1(m) \Bigg(\sum_{j=1}^{i} \mathbf{E}|\tau_j|^m + \Bigg(\sum_{j=1}^{i} \mathbf{E}\tau_j^2 \Bigg)^{m/2} \Bigg), \tag{27}$$

where $C_1(m) = \max\{(1+m/2)^m; 2(1+m/2)^{m/2} e^{1+m/2}\}$. Since $\mathbf{E}\tau_j^2 = 1$ and $\mathbf{E}|\tau_j|^m \leq \Gamma(m+1)$, from (27) we derive

$$\mathbf{E} \Big| \sum_{j=1}^{i} \tau_j \Big|^m \leq C_1(m)(1 + \Gamma(m+1)) i^{m/2}. \tag{28}$$

Substituting (28) into (26), we have

$$\mathbf{E}\|S_{n+1}(t)\|_\lambda \leq \frac{C_1^{1/m}(m)(1+\Gamma(m+1))^{1/m}}{\sqrt{n+1}} \Bigg(\sum_{i=1}^{N} i^{m/2} b_{ni} \Bigg)^{1/m} \equiv \beta_1.$$

Now, consider the case $1 \leq m < 2$. Applying the Hölder inequality twice, we obtain the following estimate:

$$\mathbf{E}\|S_{n+1}(t)\|_\lambda \leq \big(\mathbf{E}\|S_{n+1}(t)\|_\lambda^m\big)^{1/m} = \frac{1}{\sqrt{n+1}} \Bigg(\sum_{i=1}^{N} b_{ni} \mathbf{E} \Big| \sum_{j=1}^{i} \tau_j \Big|^m \Bigg)^{1/m}$$

$$\leq \frac{1}{\sqrt{n+1}} \Bigg(\sum_{i=1}^{N} b_{ni} \Bigg(\mathbf{E} \Bigg(\sum_{j=1}^{i} \tau_j \Bigg)^2 \Bigg)^{m/2} \Bigg)^{1/m} = \frac{1}{n+1} \Bigg(\sum_{i=1}^{N} i^{m/2} b_{ni} \Bigg)^{1/m} \equiv \beta_1.$$

By analogy with the above,

$$\|\eta_i(t)\|_\mu = \frac{|\tau_i|}{\sqrt{n+1}} \Bigg(\sum_{j=N+1}^{i-1} b_{nj} \Bigg)^{1/m}, \quad i = N+2, \dots, n+1,$$

$$\sum_{i=N+2}^{n+1} \mathbf{E}\|\eta_i(t)\|_\mu^k \leq k! B^2 H^{k-2}/2, \quad k = 2, 3, \dots,$$

$$\mathbf{E}\|\widetilde{S}_{n+1}(t)\|_\mu \leq \beta_2 = \frac{C(m)}{\sqrt{n+1}} \Bigg(\sum_{i=N+1}^{n} (i-N)^{m/2} b_{ni} \Bigg)^{1/m}.$$

Observe that

$$\max\{\beta_1;\beta_2\} \le \beta = C(m)(n+1)^{-1/2}\left(\sum_{i=1}^{n} i^{m/2} b_{ni}\right)^{1/m}.$$

Substituting B^2, H, and β into (25), we obtain (4). Theorem 1 is proven.

PROOF OF THEOREM 2. From (19) and Lemma 2 it follows that

$$\mathbf{E}|A_n|^r \le 2^{r-1}\left(\mathbf{E}\left|\int_0^{\frac{N+1}{n+1}} \varphi_n(t, G_n(t))\,dt\right|^r + \mathbf{E}\left|\int_{\frac{N+1}{n+1}}^{1} \varphi_n(t, G_n(t))\,dt\right|^r\right)$$

$$= 2^{r-1}\mathbf{E}\left|\int_0^{\frac{N+1}{n+1}} \varphi_n\big(t, S^0_{n+1}(t)\big)\,dt\right|^r + 2^{r-1}\mathbf{E}\left|\int_{\frac{N+1}{n+1}}^{1} \varphi_n\big(t, S^0_{n+1}(t)\big)\,dt\right|^r$$

$$\le 2^r\mathbf{E}\left|\int_0^{\frac{N+1}{n+1}} \varphi_n(t, S_{n+1}(t))\,dt\right|^r + 2^{r-1}\sqrt{3}\mathbf{E}\left|\int_{\frac{N+1}{n+1}}^{1} \varphi_n(t, -\widetilde{S}_{n+1}(t))\,dt\right|^r. \tag{29}$$

Substituting (22) and (23) into (29), we obtain

$$\mathbf{E}|A_n|^r \le 4^r\left(\sum_{i=1}^{n} a_{ni}\right)^r + 4^{r-1}\big\{2\mathbf{E}\|S_{n+1}(t)\|_\lambda^{rm} + \sqrt{3}\mathbf{E}\|\widetilde{S}_{n+1}(t)\|_\mu^{rm}\big\}.$$

It is proven in [19] that, for independent centered random variables $Y_1,\dots,Y_n$ in a separable Banach space, the following inequality holds:

$$\mathbf{E}\big|\,\|S_n\| - \mathbf{E}\|S_n\|\,\big|^l \le (\widetilde{K}l)^l\left(\sum_{i=1}^{n}\mathbf{E}\|Y_i\|^l + \left(\sum_{i=1}^{n}\mathbf{E}\|Y_i\|^2\right)^{l/2}\right), \quad l \ge 2, \tag{30}$$

where $S_n = \sum_{i=1}^n Y_i$, $\tilde{K}$ is an absolute positive constant. Putting $l = rm$ in (30), we obtain

$$\mathbf{E}\big|\,\|S_{n+1}(t)\|_\lambda - \mathbf{E}\|S_{n+1}(t)\|_\lambda\big|^{rm} \le (\widetilde{K}_1 rm)^{rm}\left\{\sum_{i=1}^{N}\mathbf{E}\|\xi_i(t)\|_\lambda^{rm} + \left(\sum_{i=1}^{N}\mathbf{E}\|\xi_i(t)\|_\lambda^2\right)^{\frac{rm}{2}}\right\},$$

$$\mathbf{E}\big|\,\|\widetilde{S}_{n+1}(t)\|_\mu - \mathbf{E}\|\widetilde{S}_{n+1}(t)\|_\mu\big|^{rm} \le (\widetilde{K}_2 rm)^{rm}\left\{\sum_{i=N+2}^{n+1}\mathbf{E}\|\eta_i(t)\|_\mu^{rm} + \left(\sum_{i=N+2}^{n+1}\mathbf{E}\|\eta_i(t)\|_\mu^2\right)^{\frac{rm}{2}}\right\}.$$

It is not difficult to verify that

$$\mathbf{E}\|\xi_i(t)\|_\lambda^{rm} \le \frac{\Gamma(rm+1)}{(n+1)^{rm/2}}\left(\sum_{j=i}^{N} b_{nj}\right)^r, \quad \mathbf{E}\|\xi_i(t)\|_\lambda^2 = \frac{1}{n+1}\left(\sum_{j=i}^{N} b_{nj}\right)^{2/m},$$

$$\mathbf{E}\|\eta_i(t)\|_\mu^{rm} \le \frac{\Gamma(rm+1)}{(n+1)^{rm/2}}\left(\sum_{j=N+1}^{i-1} b_{nj}\right)^r, \quad \mathbf{E}\|\eta_i(t)\|_\mu^2 = \frac{1}{n+1}\left(\sum_{j=N+1}^{i-1} b_{nj}\right)^{2/m}.$$

It remains to use the simple inequality (for an arbitrary norm)

$$\mathbf{E}\|S_{n+1}(t)\|^{rm} \le 2^{rm-1}\mathbf{E}|\,\|S_{n+1}(t)\| - \mathbf{E}\|S_{n+1}(t)\|\,|^{rm} + 2^{rm-1}(\mathbf{E}\|S_{n+1}(t)\|)^{rm}$$

and the upper bounds for $\mathbf{E}\|S_{n+1}(t)\|_\lambda$ and $\mathbf{E}\|\tilde{S}_{n+1}(t)\|_\mu$ obtained in Theorem 1. Theorem 2 is proven.

3.2. Proofs of Theorems 3 and 4.

Proof of the Proposition. Items (1)–(3) are immediate from the definitions. Prove item (4). Indeed, if h is a nondecreasing function, then

$$\int_{\mathbb{R}} g(t,z)|h'(t)|\,dt = \int_{-\infty}^{z} F(t)h'(t)\,dt + \int_{z}^{\infty}(1-F(t))h'(t)\,dt$$
$$= h(z)(2F(z)-1) - \int_{-\infty}^{z} h(t)\,dF(t) + \int_{z}^{\infty} h(t)\,dF(t)$$
$$\le |h(z)| + \int_{\mathbb{R}} |h(t)|\,dF(t) = |h(z)| + \mathbf{E}|h(X_1)|,$$

and the same estimate holds obviously if h is a nonincreasing function. Thus, $\alpha_k \le 2^{k-1}(\mathbf{E}|h(X_1)|^k + (\mathbf{E}|h(X_1)|)^k) \le 2^k\mathbf{E}|h(X_1)|^k$.

By the condition in item (5), finiteness of H_F is equivalent to that of $\widetilde{H}_F$. Put $P(t) = \mathbf{P}\{|X_1| \ge t\}$, $t > 0$. It is not difficult to see that the convergence of

$$\int_{\mathbb{R}} \sqrt{F(t)(1-F(t))}\,dt$$

is equivalent to that of $\int_0^\infty \sqrt{P(t)}dt$. Thus, $P(t) = o(t^{-2})$ as $t \to \infty$ (see the Remark before Theorem 3). Whence it follows that $\mathbf{E}|X_1|^2(\log^+|X_1|)^{-1-\varepsilon} < \infty$ for all $\varepsilon > 0$.

We now prove item (6). Since $|h'(t)| \le \delta|t|^\beta$, for every $t_0 > 0$ we have

$$H_F \le \delta\int_{\mathbb{R}} |t|^\beta\sqrt{F(t)(1-F(t))}\,dt \le \delta\int_{-\infty}^{0} |t|^\beta\sqrt{F(t)}\,dt + \delta\int_{0}^{\infty} t^\beta\sqrt{1-F(t)}\,dt$$
$$= \delta\int_0^\infty t^\beta\left(\sqrt{1-F(t)} + \sqrt{F(-t)}\right)\,dt \le \delta\sqrt{2}\int_0^\infty t^\beta\sqrt{1-F(t)+F(-t)}\,dt$$
$$\le \frac{\delta\sqrt{2}}{\beta+1}t_0^{\beta+1} + \delta\sqrt{2}\int_{t_0}^{\infty} t^\beta\sqrt{P(t)}\,dt.$$

Furthermore, for all $t \ge t_0$ from the Chebyshev inequality we obtain

$$P(t) \le \frac{\mathbf{E}|X_1|^{2(\beta+1)}(\log^+|X_1|)^{2+\varepsilon}}{t^{2(\beta+1)}(\log t)^{2+\varepsilon}}.$$

Thus, we have

$$H_F \le c_1 + c_2\int_{t_0}^{\infty}\frac{dt}{t(\log t)^{1+\varepsilon/2}} < \infty,$$

where c_1 are c_2 are some positive constants. The Proposition is proven.

PROOF OF THEOREM 3. Put

$$S_n(t) = \sum_{i=1}^{n} \xi_i(t), \quad \xi_i(t) = F(t) - \mathbf{I}\{X_i < t\}.$$

Obviously, $\mathbf{E}\xi_i(t) = 0$, $\mathbf{E}\xi_i^2(t) = F(t)(1 - F(t))$. Since

$$\phi_n(F(t)) - \phi_n(F_n(t)) \le nc_n|F(t) - F_n(t)| = c_n|S_n(t)|; \tag{31}$$

substituting (31) into (13), we obtain

$$L_n + \gamma_n \le c_n \int_{\mathbb{R}} |S_n(t)||h'(t)|\,dt = c_n\|S_n\|, \tag{32}$$

where $\|\cdot\|$ is the standard norm of $\mathscr{L}_1(\mathbb{R}, \mu)$, $\mu(dt) = |h'(t)|dt$.

By the definition of the norm in $\mathscr{L}_1(\mathbb{R}, \mu)$, we have

$$\|\xi_i\| = \int_{\mathbb{R}} |F(t) - \mathbf{I}\{X_i < t\}|\,|h'(t)|\,dt = \int_{\mathbb{R}} g(t, X_i)|h'(t)|\,dt.$$

Whence we obtain

$$\mathbf{E}\|\xi_i\|^k = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(t, z)|h'(t)|\,dt \right)^k dF(z) \equiv \alpha_k.$$

Now we evaluate $\mathbf{E}\|S_n\|$:

$$\begin{aligned}\mathbf{E}\|S_n\| &= \int_{\mathbb{R}} \mathbf{E}|S_n(t)|\,|h'(t)|\,dt \le \int_{\mathbb{R}} \left(\mathbf{E}S_n^2(t)\right)^{1/2}|h'(t)|\,dt \\ &= \sqrt{n} \int_{\mathbb{R}} (F(t)(1 - F(t)))^{1/2}|h'(t)|\,dt \equiv H_F\sqrt{n}.\end{aligned}$$

It is proven in [20] that, if independent random variables $Y_1, \dots, Y_n$ in a separable Banach space satisfy

$$\mathbf{P}\{\|Y_1 + \cdots + Y_n\| \ge u_0\} \le \frac{1}{24} \quad \text{and} \quad \sum_{i=1}^{n} \mathbf{E}\|Y_i\|^t/u_0^t \le \frac{1}{36}, \tag{33}$$

then the following inequality holds for all $u > u_0$:

$$\mathbf{P}\{\|Y_1 + \cdots + Y_n\| \ge u\} \le \exp\left\{ -\frac{\log 3}{2} \left(\frac{u - u_o}{2u_0} \right)^{\frac{\log 2}{\log 3}} \right\} + 6^{t+2} \frac{\sum_{i=1}^{n} \mathbf{E}\|Y_i\|^t}{2(u - u_o)^t}. \tag{34}$$

In the same paper it is noted that if the first condition in (33) holds, then both conditions of (33) hold for $u_0' = (36\sum_{i=1}^{n} \mathbf{E}\|Y_i\|^t)^{1/t}$. Relation (14) follows from (32) and (34).

Next, let $\alpha_k \le k!B^2H^{k-2}/2$, $k = 2, 3, \dots$. Then

$$\sum_{i=1}^{n} \mathbf{E}\|\xi_i\|^k = n\alpha_k \le k!(nB^2)\frac{H^{k-2}}{2}$$

and (15) follows from (25).

Suppose that $|X_1| \le b$ almost surely. Then

$$\|\xi_i\| = \int_{-b}^{X_i} F(t)|h'(t)|\,dt + \int_{X_i}^{b} (1 - F(t))|h'(t)|\,dt \le \int_{-b}^{b} |h'(t)|\,dt \equiv H_0.$$

Finally, to obtain (16) we employ inequality (1.2) in [21]. Theorem 3 is proven.

Proof of Theorem 4. From (30) and (32) we have

$$\begin{aligned}\mathbf{E}|L_n + \gamma_n|^k &\le c_n^k \mathbf{E}\|S_n\|^k \le 2^{k-1} c_n^k \{(\mathbf{E}\|S_n\|)^k + \mathbf{E}|\,\|S_n\| - \mathbf{E}\|S_n\|\,|^k\} \\ &\le 2^{k-1} c_n^k (\mathbf{E}\|S_n\|)^k + 2^{k-1} c_n^k (C_0 k)^k \left\{ \sum_{i=1}^n \mathbf{E}\|\xi_i\|^k + \left(\sum_{i=1}^n \mathbf{E}\|\xi_i\|^2 \right)^{k/2} \right\} \\ &\le 2^{k-1} c_n^k \big(H_F^k n^{k/2} + (C_0 k)^k \big(n\alpha_k + n^{k/2} \alpha_2^{k/2} \big) \big) \le 2^{k-1} c_n^k \big(H_F^k + (Ck)^k \alpha_k \big) n^{k/2},\end{aligned}$$

where C_0 and C are absolute positive constants. Theorem 4 is proven.

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