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PROBABILITY INEQUALITIES AND LIMIT THEOREMS FOR GENERALIZED L-STATISTICS

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Abstract. We obtain exponential upper bounds for tails of distributions of generalized L-statistics based on a sample from an exponential distribution. We prove the asymptotic normality of generalized L-statistics based on a sample from the uniform distribution on [0, 1] and of L-statistics with decomposed kernels (without any restrictions on the sample distribution type).

Keywords: order statistics, L-statistics, integral-type statistics, exponential upper bounds, moment inequalities, Banach spaces, asymptotic normality.

1. INTRODUCTION

Let $X_1, X_2, \dots$ be independent identically distributed random variables. We denote by $X_{n:1} \leqslant \dots \leqslant X_{n:n}$ the order statistics based on a sample $\{X_i;\ i \leqslant n\}$. Consider a linear combination of order statistics

$$L_n^{(1)} = \sum_{i=1}^{n} c_{ni} X_{n:i}$$

called a (classical) L*-statistic*. L-order statistics have numerous applications (in particular, in estimation theory). They are used, for example, in estimation of location and scale parameters (see [5], [15]). For some parametric families, the coefficients c_{ni} can be chosen so that L-statistics are, in a certain sense, equivalent to the maximum-likelihood estimates (their variances are asymptotically equivalent), which are, as a rule, optimal (see [20]).

Together with classical L-statistics, linear combinations of functions of order statistics are also used (and also called L-statistics):

$$L_n^{(2)} = \sum_{i=1}^{n} c_{ni} h(X_{n:i}),$$

where h is a measurable function called a kernel. If h is a monotone function, then the corresponding statistic $L_n^{(2)}$ clearly is representable in the form of statistic $L_n^{(1)}$ based on the sample $\{h(X_i);\ i \leqslant n\}$.

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In a majority of works devoted to the asymptotical analysis of L-statistics, there was considered the case of so-called *regular* coefficients

$$c_{ni} = n^{-1} J(i/(n+1)) \quad \text{or} \quad c_{ni} = \int_{(i-1)/n}^{i/n} J(t)\,\mathrm{d}t,$$

where J is a sufficiently smooth function (see, foe example, [1]–[3], [16], [17], [24]), or the case of *asymptotically regular* coefficients where c_{ni} are given by the formulas above with an accuracy of $\mathrm{o}(1/n)$ uniformly in all i (see [1]–[3], [24]).

All works related to the asymptotical analysis of linear combinations of functions of order statistics can be relatively divided into three groups. The first group is devoted to the analysis of statistics of the form $L_n^{(2)}$ based on a sample from an exponential distribution. This is firstly connected with the fact that, using quantile transformations, every distribution can be reduced to an arbitrary continuous (for example, an exponential or uniform) one, i.e., we can define a sample $\{Y_i;\, i \leqslant n\}$ with arbitrary distribution function G by the formula $Y_i = G^{-1}(F(X_i))$, where F is the continuous distribution function of X_1, and $G^{-1}(z) = \inf\{t\colon\ G(t) \geqslant z\}$ is the quantile transformation of the distribution function of Y_1. Since the superposition of G^{-1} and F is monotone, the statistics $G^{-1}(F(X_{n:1})) \leqslant \cdots \leqslant G^{-1}(F(X_{n:n}))$ are order statistics based on the sample $\{Y_i;\, i \leqslant n\}$. Consequently,

$$\sum_{i=1}^{n} c_{ni} h(Y_{n:i}) = \sum_{i=1}^{n} c_{ni} \tilde{h}(X_{n:i}),$$

where $\tilde{h}(x) = h(G^{-1}F((x)))$.

Secondly, this is conditioned by the convenience of analysis of the structure of order statistics that are based on samples from an exponential distribution and are partial-sum processes constructed by virtue of independent exponential random variables (for details, see Section 2). Such a representation simplifies the proofs of limit theorems for corresponding L-statistics (see, for example, [4], [11], [12]). Note that such a structure of order statistics allows one to reject the requirements of monotonicity of the kernel h and regularity of the coefficients c_{ni}.

The second group is related to the properties of order statistics based on a sample from the uniform distribution on [0, 1] (see [7], [9], [13], [16], [22]). Note that, although L-statistics constructed using the indicated distributions are represented by each other, the results obtained for these statistics do not follow from each other because of different assumptions on the kernels.

The third group is devoted to investigation of statistics of the form $L_n^{(2)}$ without any additional restrictions on the distribution of a sample. In this group, one mostly requires the monotonicity of the kernel H, which, as mentioned before, is equivalent to the investigation of statistics $L_n^{(1)}$. In this relation, we refer to [1]–[3], [9], [17], [18], [23], [24].

The most general L-statistics found in the literature are additive functionals of order statistics of the form

$$\Phi_n = \sum_{i=1}^{n} h_{ni}(X_{n:i}), \tag{1}$$

where $h_{ni}\colon \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, are measurable functions. In particular, if $h_{ni}(y) = c_{ni} h(y)$, we get the class of statistics $L_n^{(2)}$, which we call the class *with decomposed kernels*. If, moreover, h is a monotone function, then, as mentioned above, we get statistics of the form $L_n^{(1)}$.

In the generality considered, it is natural to call functionals of the form (1) *generalized L-statistics*. They were first introduced in [27], [28], where asymptotic expansions for distributions of some partial forms of these statistics were obtained. The Fourier analysis of the distributions of Φ_n is given in [7]. Note that integral statistics (integral functionals of empirical distribution functions of, for example, the Cramér–Anderson–Darling statistic) are representable in the form (1) but not in the form of classical L-statistics (for more details, see [7], [27]). Also note that the term "generalized" for L-statistics was introduced in [21], where a generalization of the theory of classical L-statistics was considered in another direction of construction of order statistics.

In [9], we earlier obtained exponential upper bounds for tails of distributions of statistics of the form (1) based on a sample from the uniform distribution on [0, 1] and also moment inequalities for them; we also obtained similar upper bounds for linear combinations of functions of order statistics (L-statistics with decomposed kernels) without any additional restrictions on the distribution of a sample and without the requirements of monotonicity of the kernel and regular representation of the coefficients c_{ni}. In this paper, we obtain exponential upper bounds for tails of the distribution of statistics of the form (1) based on a sample from an exponential distribution. Note that, in [8], moment inequalities for such statistics are obtained. We also prove a limit theorem for a class of generalized L-statistics constructed by virtue of a collection of centered order statistics when the limit distribution is an integral functional of some Gaussian process. Moreover, in this paper, we investigate the asymptotic normality of generalized L-statistics based on a sample from the uniform distribution on [0, 1] and that of L-statistics with decomposed kernels (without any restriction on the sample distribution type). In particular, for the asymptotic normality of the latter, we essentially relaxed the restrictions on weights assumed in [17] that actually mean the asymptotic regularity of the coefficients c_{ni}. Conditions on c_{ni} proposed in this paper are weaker than this requirement.

2. PROBABILITY INEQUALITIES

In this section, we consider samples from the exponential distribution with parameter 1 only. Consider the generalized L-statistics

$$\overline{\Phi}_n = \sum_{i=1}^{n} h_{ni}(X_{n:i} - \mathbb{E}X_{n:i}). \tag{2}$$

THEOREM 1. *Let the functions $\{h_{ni};\ i \leqslant n\}$ in* (2) *satisfy the following condition on* $\mathbb{R}$:

$$|h_{ni}(x)| \leqslant a_{ni} + b_{ni}|x|^m \quad \text{for some } m \geqslant 1, \tag{3}$$

where a_{ni} and b_{ni} are positive constants depending on i and n only. Then

$$\mathbb{P}\{\overline{\Phi}_n \geqslant y\} \leqslant \exp\left\{-\frac{(y-\Lambda)^{2/m} - 2\beta y^{1/m}}{2(B^2 + Hy^{1/m})}\right\}, \tag{4}$$

where

$$\Lambda = \sum_{i=1}^{n} a_{ni}, \qquad B^2 = \sum_{i=1}^{n} (n+1-i)^{-2} B_{ni}^{2/m},$$

$$H = \max_{1\leqslant i\leqslant n} (n+1-i)^{-1} B_{ni}^{1/m}, \quad B_{ni} = \sum_{j=i}^{n} b_{nj},$$

$$\beta = \begin{cases} \left(\sum\limits_{i=1}^{n} b_{ni}\left(\sum\limits_{j=1}^{i}(n+1-j)^{-2}\right)^{m/2}\right)^{1/m} & \text{if } 1 \leqslant m < 2, \\ \left(\sum\limits_{i=1}^{n}(n+1-i)^{-2}\sum\limits_{j=i}^{n} b_{nj}^{2/m}\right)^{1/2} & \text{if } m \geqslant 2. \end{cases}$$

Let us consider the partial case $h_{ni}(x) = x$. Then $a_{ni} = 0$, $b_{ni} = 1$, $m = 1$, and the statistic $\overline{\Phi}_n$ is of the form

$$\overline{\Phi}_n = \sum_{i=1}^{n} (X_i - 1),$$

where $X_1, \ldots, X_n$ are independent exponential random variables with parameter 1; we also have

$$B^2 = n, \qquad H = 1, \qquad \beta \leqslant (2/n)^{1/2} \sum_{k=1}^{n} \left(\frac{k}{n+1-k} \right)^{1/2} \leqslant C\sqrt{n}.$$

In this case, the classical Bernstein inequality gives the estimate

$$\mathbb{P}\{\overline{\Phi}_n \geqslant y\sqrt{n}\} \leqslant \exp\left\{ -\frac{y^2}{2(1+yn^{-1/2})} \right\}, \quad y \geqslant 0,$$

while inequality (4) yields the estimate

$$\mathbb{P}\{\overline{\Phi}_n \geqslant y\sqrt{n}\} \leqslant \exp\left\{ -\frac{y^2 - 2Cy}{2(1+yn^{-1/2})} \right\}, \quad y \geqslant 2C.$$

The comparison of these two inequalities allows one to make a rather unexpected conclusion that, in the case considered, the deviation probabilities of two sums

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - 1) \quad \text{and} \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |X_{n:i} - \mathbb{E}X_{n:i}|$$

are essentially the same.

Let us introduce the centered generalized L-statistic

$$\widetilde{\Phi}_n = \sum_{i=1}^{n} h_{ni}(X_{n:i}) - \sum_{i=1}^{n} h_{ni}(\mathbb{E}X_{n:i}). \tag{5}$$

As a direct corollary of Theorem 1, one can obtain the following:

THEOREM 2. *Let the functions $\{h_{ni};\ i \leqslant n\}$ in (5) be continuously differentiable on $[0, \infty)$ and such that, for all $x \geqslant 0$,*

$$\left|h'_{ni}(x)\right| \leqslant \alpha_{ni} + \beta_{ni} x^p \quad \textit{for some } p \geqslant 0, \tag{6}$$

where α_{ni} and β_{ni} are positive functions depending on i and n only. Then we have

$$\mathbb{P}\{\widetilde{\Phi}_n \geqslant y\} \leqslant \exp\left\{ -\frac{y^2 - 4\beta_1 y}{8(B_1^2 + H_1 y)} \right\} + \exp\left\{ -\frac{y^{2/(p+1)} - 4\beta_2 y^{1/(p+1)}}{8(B_2^2 + H_2 y^{1/(p+1)})} \right\}, \tag{7}$$

where

$$\beta_1 = \sum_{i=1}^{n} \gamma_{ni} \left(\sum_{j=1}^{i} (n+1-j)^{-2} \right)^{m/2},$$

$$\gamma_{ni} = \alpha_{ni} + c_p \beta_{ni} \left(\sum_{j=1}^{i} (n+1-j)^{-1} \right)^{p},$$

$$B_1^2 = 2 \sum_{i=1}^{n} \widetilde{B}_{ni}^2, \qquad H_1 = \max_{1 \leqslant i \leqslant n} \widetilde{B}_{ni},$$

$$\widetilde{B}_{ni} = \frac{1}{n+1-i}\sum_{j=i}^{n}\gamma_{nj}, \qquad c_p = \max\{1, 2^{p-1}\},$$

$$B_2^2 = 2\sum_{i=1}^{n}\overline{B}_{ni}^2, \qquad H_2 = \max_{1\leqslant i\leqslant n}\overline{B}_{ni},$$

$$\overline{B}_{ni} = \frac{1}{n+1-i}\left(\sum_{j=i}^{n}\beta_{nj}\right)^{1/(p+1)},$$

$$\beta_2 = \begin{cases} \left(\sum\limits_{i=1}^{n}\beta_{ni}\left(\sum\limits_{j=1}^{i}(n+1-j)^{-2}\right)^{(p+1)/2}\right)^{1/(p+1)} & \text{if } 0 \leqslant p < 1, \\ \left(\sum\limits_{i=1}^{n}(n+1-i)^{-2}\sum\limits_{j=i}^{n}\beta_{nj}^{2/(p+1)}\right)^{1/2} & \text{if } p \geqslant 1. \end{cases}$$

Proof Theorem 1. We denote $\tau_{ni} = X_{n:i} - X_{n:i-1}$, $X_{n:0} = 0$. It is known [26] that $\tau_{n1}, \ldots, \tau_{nn}$ are independent and exponentially distributed with corresponding parameters:

$$P(\tau_{ni} > t) = \mathrm{e}^{-(n+1-i)t}, \quad i = 1, \ldots, n.$$

Consequently, τ_{ni} has the same distribution as that of $(n+1-i)^{-1}Z_i$, where $Z_1, \ldots, Z_n$ ere independent exponential random variables with parameter 1. Thus, order statistics $X_{n:k}$ are representable in the form of partial sums of the indicated random variables:

$$X_{n:k} = \sum_{i=1}^{k}\frac{Z_i}{n+1-i}, \quad k = 1, \ldots, n.$$

Consider the random process

$$S_n(t) = \sum_{i=1}^{n}\xi_i(t), \quad 0 \leqslant t \leqslant 1, \tag{8}$$

where

$$\xi_i(t) = \frac{Z_i - 1}{n+1-i}\mathbb{I}\{(i-1)/n \leqslant t\}. \tag{9}$$

We define the function

$$\varphi_n(t, x) = nh_{ni}(x) \quad \text{for all } t \in [(i-1)/n, i/n),\ i = 1, \ldots, n. \tag{10}$$

The following relation holds:

$$\overline{\Phi}_n = \int_0^1 \varphi_n(t, S_n(t))\,\mathrm{d}t. \tag{11}$$

From (3) it follows that, for all $t \in [(i-1)/n, i/n)$, $i = 1, \ldots, n$, we have

$$|\varphi_n(t, x)| \leqslant na_{ni} + nb_{ni}|x|^m. \tag{12}$$

Now from (11) and (12) we get

$$
\begin{aligned}
\left|\overline{\Phi}_n\right| &\leqslant \sum_{i=1}^{n} \int\limits_{(i-1)/n}^{i/n} \left(na_{ni} + nb_{ni}\,|S_n(t)|^m\right) \mathrm{d}t \\
&= \Lambda + \int\limits_0^1 |S_n(t)|^m \lambda(\mathrm{d}t) = \Lambda + \|S_n\|^m,
\end{aligned} \tag{13}
$$

where $\lambda(\mathrm{d}t) = q(t)\,\mathrm{d}t$ and $q(t) = nb_{ni}$ for $(i-1)/n \leqslant t < i/n$, and $\|\cdot\|$ is the standard norm in the space $\mathcal{L}_m = \mathcal{L}_m([0,1],\lambda)$. Consequently,

$$
\mathbb{P}\left\{\overline{\Phi}_n \geqslant y\right\} \leqslant \mathbb{P}\left\{\|S_n\| \geqslant (y-\Lambda)^{1/m}\right\}. \tag{14}
$$

In [19], it is proved that if, for independent random variables $Y_1, \ldots, Y_n$ with values in a separable Banach space, the inequalities

$$
\sum_{j=1}^{n} \mathbb{E}\|Y_j\|^k \leqslant k!B^2H^{k-2}/2, \quad k = 2, 3, \ldots, \tag{15}
$$

hold for some constants B^2 and $H > 0$, then

$$
\mathbb{P}\left(\|Y_1 + \cdots + Y_n\| - \beta \geqslant x\right) \leqslant \exp\left\{-\frac{x^2}{2(B^2 + xH)}\right\},
$$

where $\beta = \mathbb{E}\|Y_1 + \cdots + Y_n\|$. From this we easily obtain that

$$
\mathbb{P}\left(\|Y_1 + \cdots + Y_n\| \geqslant x\right) \leqslant \exp\left\{-\frac{x^2 - 2\beta x}{2(B^2 + xH)}\right\}. \tag{16}
$$

Note that $\xi_1, \ldots, \xi_n$ are independent nonidentically distributed random variables with zero mean and values in the separable Banach space $\mathcal{L}_m$. Let us show that the random variables $\xi_1, \ldots, \xi_n$ satisfy condition (15). By the definition of the $\mathcal{L}_m$ norm we have

$$
\begin{aligned}
\|\xi_i\|^k &= \left(\int\limits_0^1 |\xi_i(t)|^m \lambda(\mathrm{d}t)\right)^{k/m} = \frac{|Z_i - 1|^k}{(n+1-i)^k}\left(\int\limits_{(i-1)/n}^{1} q(t)\,\mathrm{d}t\right)^{k/m} \\
&= \frac{|Z_i - 1|^k}{(n+1-i)^k}\left(\sum_{j=i}^{n} b_{nj}\right)^{k/m} = \frac{|Z_i - 1|^k}{(n+1-i)^k} B_{ni}^{k/m}, \quad i = 1, \ldots, n.
\end{aligned}
$$

From this we obtain

$$
\mathbb{E}\|\xi_i\|^k \leqslant \frac{k!}{2}\frac{B_{ni}^{k/m}}{(n+1-i)^k}, \quad k = 2, 3, \ldots,\ i = 1, \ldots, n,
$$

since $\mathbb{E}|Z_i - 1|^2 = 1$ and

$$\mathbb{E}|Z_i - 1|^k = \int_0^\infty |x-1|^k \mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{\mathrm{e}}\int_0^1 x^k \mathrm{e}^x\,\mathrm{d}x + \frac{1}{\mathrm{e}}\int_0^\infty x^k \mathrm{e}^x\,\mathrm{d}x$$
$$\leqslant 1/(k+1) + k!/\mathrm{e} \leqslant k!/2, \quad k \geqslant 3.$$

One easily checks that

$$\sum_{i=1}^n \mathbb{E}\|\xi_i\|^k \leqslant \frac{k!}{2}\left(\sum_{i=1}^n \frac{B_{ni}^{2/m}}{(n+1-i)^2}\right)\left(\max_{1\leqslant i\leqslant n}\frac{B_{ni}^{1/m}}{n+1-i}\right)^{k-2}$$
$$= k!B^2H^{k-2}/2, \quad k = 2, 3, \ldots .$$

Let us now estimate $\mathbb{E}\|S_n\|$. Let $m \geqslant 2$. Then we have

$$\|S_n\|^m = \int_0^1 \left|\sum_{i=1}^n \xi_i(t)\right|^m \lambda(\mathrm{d}t) = \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left|\sum_{i=1}^k \frac{Z_i - 1}{n+1-i}\right|^m nb_{nk}\,\mathrm{d}t$$
$$= \sum_{k=1}^n b_{nk}\left|\sum_{i=1}^k \frac{Z_i - 1}{n+1-i}\right|^m .$$

From this we have

$$\|S_n\|^2 = \left\{\sum_{k=1}^n b_{nk}\left|\sum_{i=1}^k \frac{Z_i-1}{n+1-i}\right|^m\right\}^{2/m} \leqslant \sum_{k=1}^n b_{nk}^{2/m}\left(\sum_{i=1}^k \frac{Z_i-1}{n+1-i}\right)^2 .$$

Consequently,

$$\mathbb{E}\|S_n\| \leqslant \left(\mathbb{E}\|S_n\|^2\right)^{1/2} \leqslant \left\{\sum_{k=1}^n b_{nk}^{2/m}\mathbb{E}\left(\sum_{i=1}^k \frac{Z_i-1}{n+1-i}\right)^2\right\}^{1/2}$$
$$= \left\{\sum_{k=1}^n b_{nk}^{2/m}\sum_{i=1}^k \frac{1}{(n+1-i)^2}\right\}^{1/2} = \left\{\sum_{i=1}^n \frac{1}{(n+1-i)^2}\sum_{k=i}^n b_{nk}^{2/m}\right\}^{1/2} \equiv \beta.$$

Let now $1 \leqslant m < 2$. Then, using the Hölder inequality twice, we get

$$\begin{aligned}\mathbb{E}\|S_n\| \leqslant \left(\mathbb{E}\|S_n\|^m\right)^{1/m} &= \left\{\sum_{k=1}^n b_{nk}\mathbb{E}\left|\sum_{i=1}^k \frac{Z_i-1}{n+1-i}\right|^m\right\}^{1/m} \\ &\leqslant \left\{\sum_{k=1}^n b_{nk}\left(\mathbb{E}\left(\sum_{i=1}^k \frac{Z_i-1}{n+1-i}\right)^2\right)^{m/2}\right\}^{1/m} \\ &= \left\{\sum_{k=1}^n b_{nk}\left(\sum_{i=1}^k \frac{1}{(n+1-i)^2}\right)^{m/2}\right\}^{1/m} \equiv \beta.\end{aligned}$$

Substituting the estimates obtained into (14) and (16), we obtain inequality (4). The theorem is proved.

Proof of Theorem 2. Using the Taylor formula with the remainder in the integral form, we get

$$\widetilde{\Phi}_n = \sum_{i=1}^n (X_{n:i} - \mathbb{E}X_{n:i}) \int_0^1 h'_{ni}\big(\mathbb{E}X_{n:i} + \theta(X_{n:i} - \mathbb{E}X_{n:i})\big)\mathrm{d}\theta.$$

From condition (6) we get the estimate

$$\begin{aligned}\left|\widetilde{\Phi}_n\right| &\leqslant \sum_{i=1}^n |X_{n:i} - \mathbb{E}X_{n:i}| \int_0^1 \left(\alpha_{ni} + \beta_{ni}|\mathbb{E}X_{n:i} + \theta(X_{n:i} - \mathbb{E}X_{n:i})|^p\right)\mathrm{d}\theta \\ &\leqslant \sum_{i=1}^n |X_{n:i} - \mathbb{E}X_{n:i}| \int_0^1 \left(\alpha_{ni} + c_p\beta_{ni}(\mathbb{E}X_{n:i})^p + c_p\beta_{ni}|X_{n:i} - \mathbb{E}X_{n:i}|^p\theta^p\right)\mathrm{d}\theta \\ &\leqslant \sum_{i=1}^n \left(\alpha_{ni} + c_p\beta_{ni}(\mathbb{E}X_{n:i})^p\right)|X_{n:i} - \mathbb{E}X_{n:i}| + c_p\sum_{i=1}^n \beta_{ni}|X_{n:i} - \mathbb{E}X_{n:i}|^{p+1} \\ &= \sum_{i=1}^n \gamma_{ni}|X_{n:i} - \mathbb{E}X_{n:i}| + c_p\sum_{i=1}^n \beta_{ni}|X_{n:i} - \mathbb{E}X_{n:i}|^{p+1}.\end{aligned} \tag{17}$$

Let

$$R_{n1} = \sum_{i=1}^n \gamma_{ni}|X_{n:i} - \mathbb{E}X_{n:i}| \quad \text{and} \quad R_{n2} = \sum_{i=1}^n \beta_{ni}|X_{n:i} - \mathbb{E}X_{n:i}|^{p+1}. \tag{18}$$

Then

$$\mathbb{P}\left\{\widetilde{\Phi}_n \geqslant y\right\} \leqslant \mathbb{P}\{R_{n1} \geqslant y/2\} + \mathbb{P}\left\{R_{n2} \geqslant y/(2c_p)\right\}. \tag{19}$$

Applying the estimates of Theorem 2 to R_{n1} and R_{n2} in (19), we get (7). The theorem is proved.

3. NORMAL APPROXIMATION

3.1. A limit theorem for the statistics A_n

Consider the generalized L-statistics

$$A_n = \sum_{i=1}^{n} h_{ni}\big(U_{n:i} - i/(n+1)\big) \tag{20}$$

based on a sample from the uniform distribution on [0, 1]. Let $w^0(t)$, $0 \leqslant t \leqslant 1$, be a Gaussian random process with zero mean and correlation function $\min\{s, t\} - st$, $0 \leqslant s, t \leqslant 1$. By $\xrightarrow{\mathrm{d}}$ we denote the weak convergence of distributions.

THEOREM 3. *Suppose that there exists a continuous function $\varphi(t, x)$, $0 \leqslant t \leqslant 1$, $x \in \mathbb{R}$, such that, for all $x \in \mathbb{R}$,*

$$\max_{1\leqslant i\leqslant n} \sup_{(i-1)/n<t\leqslant i/n} \big|nh_{ni}(x/\sqrt{n}) - \varphi(t, x)\big| \leqslant \varepsilon_n \psi(x), \tag{21}$$

where $\varepsilon_n \to 0$ as $n \to \infty$, and $\psi(x) \geqslant 0$ is a continuous function. Then

$$A_n \xrightarrow{\mathrm{d}} \int_0^1 \varphi(t, w^0(t))\,\mathrm{d}t \quad \textit{as} \quad n \to \infty. \tag{22}$$

Proof of Theorem 3. We define the random process $G_n(t)$ and function $\varphi_n(t, x)$ as follows. For all $t \in ((i-1)/n, i/n]$, $i = 1, \ldots, n$, we set

$$G_n(t) = n^{1/2}\big(U_{n:i} - i/(n+1)\big), \qquad \varphi_n(t, x) = nh_{ni}\big(xn^{-1/2}\big). \tag{23}$$

Then we have

$$A_n = \int_0^1 \varphi_n(t, G_n(t))\,\mathrm{d}t. \tag{24}$$

From (24) we get

$$A_n = \int_0^1 \varphi(t, G_n(t))\,\mathrm{d}t + \int_0^1 \{\varphi_n(t, G_n(t)) - \varphi(t, G_n(t))\}\,\mathrm{d}t. \tag{25}$$

It is known (see, for example, [6]), that, as $n \to \infty$, the distributions of the processes $G_n(t)$ converge to a Brownian bridge $w^0(t)$ in the sense of the so-called C-convergence in $D[0, 1]$. Since the function $\varphi(t, x)$ is continuous in both variables, the functional

$$T(x) = \int_0^1 \varphi(t, x(t))\,\mathrm{d}t$$

on $D[0, 1]$ is continuous in the uniform metric. Consequently, by the invariance principle (see [10]), the distributions of

$$T(G_n) = \int_0^1 \varphi(t, G_n(t))\,\mathrm{d}t$$

weakly converge to the distribution of

$$T(w^0) = \int_0^1 \varphi(t, w^0(t))\,\mathrm{d}t.$$

Let us estimate the second summand on the right-hand side of (25). From (21) it follows that

$$\int_0^1 |\varphi_n(t, G_n(t)) - \varphi(t, G_n(t))|\,\mathrm{d}t \leqslant \varepsilon_n \int_0^1 \psi(G_n(t))\,\mathrm{d}t. \tag{26}$$

By the continuity of $\psi(x)$, similarly to the above, we have

$$\int_0^1 \psi(G_n(t))\,\mathrm{d}t \xrightarrow{\mathrm{d}} \int_0^1 \psi(w^0(t))\,\mathrm{d}t \quad \text{as } n \to \infty.$$

Since $\varepsilon_n \to 0$, the right-hand side of (26) converges to zero in probability. Hence, (22) follows. The theorem is proved.

3.2. Asymptotic normality

3.2.1. Generalized L-statistics. Consider the centered generalized L-statistic

$$\bar{A}_n = \sum_{i=1}^n h_{ni}(U_{n:i}) - \sum_{i=1}^n h_{ni}(\mathbb{E}U_{n:i}), \tag{27}$$

which is also based on a sample from the uniform distribution on [0, 1] and has smooth kernels.

For all $t \in ((i-1)/n, i/n]$, $i = 1, \ldots, n$, we set

$$\alpha_n(t) = n^{1/2} h'_{ni}(i/(n+1)).$$

We denote

$$\sigma_n^2 = \int_0^1 \int_0^1 \alpha_n(x)\alpha_n(y)\big(\min\{x, y\} - xy\big)\,\mathrm{d}x\,\mathrm{d}y.$$

Since σ_n^2 is the second moment of some Gaussian random variable (see the proof of Theorem 4), we have that $\sigma_n^2 \geqslant 0$. Note that if $h'_{ni}(i/(n+1)) \neq 0$ at least for one i, then $\sigma_n^2 > 0$.

We also denote by $\mathbb{N}(0,1)$ a standard normal random variable.

THEOREM 4. *Let the functions* $\{h_{ni};\ i \leqslant n\}$ *in* (27) *be continuously differentiable in* [0, 1] *and satisfy the following conditions*:

$$\big|h'_{ni}(x) - h'_{ni}(y)\big| \leqslant b_{ni}|x - y|^\alpha, \quad 0 < \alpha \leqslant 1, \tag{28}$$

$$\sum_{i=1}^n b_{ni} = \mathrm{o}\left(n^{(\alpha+1)/2}\sigma_n\right), \tag{29}$$

$$\sum_{i=1}^n \big|h'_{ni}(i/(n+1))\big| = \mathrm{o}\big(n\sigma_n(\ln n)^{-1}\big). \tag{30}$$

Then

$$\sigma_n^{-1}\bar{A}_n \xrightarrow{\mathrm{d}} \mathbb{N}(0,1) \quad \text{as} \quad n \to \infty. \tag{31}$$

Remark. Let us consider the partial case

$$h_{ni}(x) = c_{ni}h(x), \quad \big|h'(x) - h'(y)\big| \leqslant K|x-y|^{\alpha}, \quad 0 < \alpha \leqslant 1.$$

Let also $h'(0) = 0$. Then, in the case $0 < \alpha < 1$, condition (29) implies condition (30), and, in the case $\alpha = 1$, for conditions (29) and (30) to be satisfied, it suffices that

$$\sum_{i=1}^{n} |c_{ni}| = \mathrm{o}\big(n\sigma_n(\ln n)^{-1}\big).$$

Note also that, in Theorem 4, one does not require the uniform attraction of the step function $\alpha_n(t)$ to some continuous function, i.e., a condition similar to (21), while, for example, in [17], such a condition is essential.

We further set, in (27), $h_{nk}(x) = xn^{1/2}$, $h_{ni}(x) = 0$, $i \neq k$, where $k/n \to p$. Then $b_{ni} = 0$, $i \leqslant n$, $\bar{A}_n = \sqrt{n}(U_{n:k} - \mathbb{E}U_{n:k})$, and

$$\sigma_n^2 = n^2 \int\limits_{(k-1)/n}^{k/n} \int\limits_{(k-1)/n}^{k/n} \big(\min\{x,y\} - xy\big)\,\mathrm{d}x\,\mathrm{d}y \longrightarrow p(1-p) \quad \text{as } n \to \infty.$$

Consequently, all conditions of Theorem 4 are satisfied, i.e., relation (31) holds. Since $\mathbb{E}U_{n:k} = k/(n+1) \to p$ and $\sigma_n \to \sqrt{p(1-p)}$ as $n \to \infty$, from (31) we get the well-known result (see, for example, [25]):

COROLLARY. *Let $k = k(n) \to \infty$ and $k/n \to p$, $0 < p < 1$, as $n \to \infty$. Then we have*

$$\frac{\sqrt{n}(U_{n:k} - p)}{\sqrt{p(1-p)}} \xrightarrow{\mathrm{d}} \mathbb{N}(0,1).$$

Proof of Theorem 4. As shown in [6], there exists a representation of the processes $G_n(t)$ (23) on a probability space with the process $w^0(t)$ such that, for all $x > 0$,

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant 1} \big|G_n(t) - w^0(t)\big| \geqslant n^{-1/2}(C_1 \ln n + x)\Big) \leqslant K\mathrm{e}^{-C_2 x}, \tag{32}$$

where C_1, C_2, and K are absolute positive constants. We further suppose that the processes $G_n(t)$ and $w^0(t)$ are defined on the same probability space by the method of [6].

We denote

$$R_n = \bar{A}_n - \sum_{i=1}^{n} h'_{ni}(i/(n+1))(U_{n:i} - \mathbb{E}U_{n:i}). \tag{33}$$

We have

$$\begin{aligned}
\sigma_n^{-1}\bar{A}_n &= \sigma_n^{-1}\sum_{i=1}^{n} h'_{ni}(i/(n+1))(U_{n:i} - \mathbb{E}U_{n:i}) + \sigma_n^{-1}R_n \\
&= \int\limits_0^1 \sigma_n^{-1}\alpha_n(t)G_n(t)\,\mathrm{d}t + \sigma_n^{-1}R_n
\end{aligned}$$

$$= \int_0^1 \sigma_n^{-1}\alpha_n(t)w^0(t)\,\mathrm{d}t + \int_0^1 \sigma_n^{-1}\alpha_n(t)\left\{G_n(t) - w^0(t)\right\}\,\mathrm{d}t + \sigma_n^{-1}R_n. \tag{34}$$

Let us estimate R_n. By (28) we have the estimate

$$|R_n| \leqslant \sum_{i=1}^{n} b_{ni}|U_{n:i} - \mathbb{E}U_{n:i}|^{\alpha+1}.$$

From (29) and the inequality $\mathbb{E}|U_{n:i} - \mathbb{E}U_{n:i}|^{\alpha+1} \leqslant (\mathbb{D}U_{n:i})^{(\alpha+1)/2} \leqslant n^{-(\alpha+1)/2}$ it follows that $\sigma_n^{-1}\mathbb{E}|R_n| \to 0$ as $n \to \infty$. Therefore, $\sigma_n^{-1}R_n \underset{\mathrm{p}}{\longrightarrow} 0$ as $n \to \infty$, where by $\underset{\mathrm{p}}{\longrightarrow}$ we denote the convergence in probability. Consider the second summand on the right-hand side of (34). From (32) we get that, for every $\beta > 0$, there exists a positive constant $C(\beta)$ such that

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant 1}\left|G_n(t) - w^0(t)\right| \geqslant C(\beta)n^{-1/2}\ln n\Big) \leqslant Kn^{-\beta}.$$

We set

$$\varepsilon_n = C(\beta)\frac{\ln n}{n\sigma_n}\sum_{i=1}^{n}\left|h'_{ni}(i/(n+1))\right|.$$

By (30) we have that $\varepsilon_n \to 0$ as $n \to \infty$. From this we find

$$\begin{aligned}
&\mathbb{P}\left\{\int_0^1 \sigma_n^{-1}|\alpha_n(t)|\left|G_n(t) - w^0(t)\right|\,\mathrm{d}t \geqslant \varepsilon_n\right\}\\
&\quad\leqslant \mathbb{P}\left\{\sup_{0\leqslant t\leqslant 1}\left|G_n(t) - w^0(t)\right|\int_0^1 \sigma_n^{-1}|\alpha_n(t)|\,\mathrm{d}t \geqslant \varepsilon_n\right\}\\
&\quad\leqslant \mathbb{P}\Big(\sup_{0\leqslant t\leqslant 1}\left|G_n(t) - w^0(t)\right| \geqslant C(\beta)n^{-1/2}\ln n\Big) \leqslant Kn^{-\beta}.
\end{aligned}$$

Consequently,

$$\int_0^1 \sigma_n^{-1}\alpha_n(t)\left\{G_n(t) - w^0(t)\right\}\,\mathrm{d}t \underset{\mathrm{p}}{\longrightarrow} 0, \quad n \to \infty.$$

We denote

$$\eta_n = \int_0^1 \sigma_n^{-1}\alpha_n(t)w^0(t)\,\mathrm{d}t.$$

It is clear that η_n is normally distributed (as a linear continuous functional of a Gaussian process). Clearly, $\mathbb{E}\eta_n = 0$. Let us find $\mathbb{E}\eta_n^2$. Since $\mathbb{E}w^0(x)w^0(y) = \min\{x, y\} - xy$, by the Fubini theorem and definition of σ_n we get

$$\mathbb{E}\eta_n^2 = \frac{1}{\sigma_n^2}\mathbb{E}\int_0^1\int_0^1 \alpha_n(x)\alpha_n(y)w^0(x)w^0(y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{\sigma_n^2}\int_0^1\int_0^1 \alpha_n(x)\alpha_n(y)\mathbb{E}w^0(x)w^0(y)\,\mathrm{d}x\,\mathrm{d}y = 1.$$

Thus, for every $n \geqslant 1$, η_n is a standard normal random variable. The theorem is proved.

3.2.2. L-statistics with decomposed kernels. Consider L-statistics of the form

$$L_n = \sum_{i=1}^n c_{ni}h(X_{n:i}). \tag{35}$$

As in [9], we do not require the monotonicity of the function h and any additional assumptions on the distribution of a sample, i.e., X_1 has an arbitrary distribution function F. Throughout this section, we suppose that h is a left-continuous function with finite variation on finite intervals. We denote by $|\mathrm{d}h(t)|$ the total-variation measure generated by h. In what follows, the existence of an integral $\int f\,\mathrm{d}h$ always means the finiteness of $\int |f||\mathrm{d}h|$.

We define the function $\varphi_n(x)$ by the relation

$$\varphi_n(x) = \int_0^x c_n(t)\,\mathrm{d}t,$$

where

$$c_n(t) = nc_{ni}, \quad t \in ((i-1)/n, i/n],\ i = 1, \dots, n, \quad c_n(0) = nc_{n1}.$$

One easily sees that $\varphi_n(x)$ is a continuous piecewise-linear function and that

$$\varphi_n(0) = 0, \qquad \varphi_n(k/n) = \sum_{i=1}^k c_{ni}, \qquad k = 1, \dots, n;$$

in addition the statistic L_n admits the representation (see [9])

$$L_n = \int_{\mathbb{R}} h(t)\,\mathrm{d}\varphi_n(F_n(t)),$$

where $F_n(t)$ is the empirical distribution function based on the sample $X_1, \dots, X_n$.

We introduce the following notation (under the condition of existence of the corresponding integrals):

$$\mu_n = \int_{\mathbb{R}} h(t)\,\mathrm{d}\varphi_n(F(t)), \qquad \tilde{c}_n = \max_{1\leqslant i\leqslant n-1}|c_{n,i+1} - c_{ni}|,$$

$$\sigma_n^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} c_n(F(x))c_n(F(y))\big(\min\{F(x), F(y)\} - F(x)F(y)\big)\,\mathrm{d}h(x)\,\mathrm{d}h(y),$$

$$Y_{n1} = \int_{\mathbb{R}} c_n(F(t))\bigl(F(t) - \mathbb{I}\{X_1 < t\}\bigr)\,\mathrm{d}h(t).$$

THEOREM 5. *Suppose that* $0 < \sigma_n < \infty$ *for all* n *and that, moreover,*

$$\mathbb{E}Y_{n1}^2\mathbb{I}\left\{|Y_{n1}| \geqslant \varepsilon\sigma_n\sqrt{n}\right\} = \mathrm{o}(\sigma_n^2) \quad \textit{for all } \varepsilon > 0, \tag{36}$$

$$\tilde{c}_n = \mathrm{o}\bigl(n^{-3/2}\sigma_n\bigr), \tag{37}$$

$$\int_{\mathbb{R}} F(t)(1 - F(t))|\mathrm{d}h(t)| < \infty. \tag{38}$$

Then

$$\frac{\sqrt{n}(L_n - \mu_n)}{\sigma_n} \xrightarrow{\mathrm{d}} \mathbb{N}(0, 1) \quad \textit{as} \quad n \to \infty. \tag{39}$$

Remark. In [17], the asymptotic normality of statistics of the form (35) was also studied; in particular, it was shown that relation (39) holds under some moment restrictions and the following additional hypotheses:

(i) the sequence of functions $c_n(t)$ converges to a bounded function uniformly on [0, 1];

(ii) There exists c such that $h(c) = 0$.

In [17], a relation similar to (39) for L-statistics (35) with regular weights was also obtained. In Theorem 5, the conditions on c_{ni} and h are weaker. Note that condition (i) means that the weights c_{ni} are asymptotically regular. Note also that condition (ii) is an additional restriction on c_{ni}. If $h(x) > 0$ on the support of the distribution F (obviously, in this case, condition (ii) is not satisfied), then, for the statement of [17] to be true, it is necessary that $\sum_{i=1}^{n} c_{ni} = \mathrm{o}(n^{-1/2}\sigma_n)$.

In the case of a regular representation of the coefficients c_{ni}, we have the following relations:

$$\sigma_n \sim \sigma, \qquad \tilde{c}_n = \mathrm{O}\bigl(n^{-2}\bigr), \qquad |Y_{n1}| \leqslant C\xi,$$

where σ is obtained from σ_n by replacing $c_n(t)$ by $J(t)$, and J is a Lipschitz function in the representation of c_{ni}, $\xi = \int_{\mathbb{R}} g(t, X_1)|\mathrm{d}h(t)|$, $g(t, z) = F(t)$ for $t \leqslant z$ and $g(t, z) = 1 - F(t)$ for $t > z$. In this case, (37) is automatically satisfied, and (36) is satisfied if $\mathbb{E}\xi^2 < \infty$.

Without the requirement of regularity of the coefficients c_{ni}, one can easily construct an example where (36)–(38) are satisfied but the sequence $c_n(t)$ does not converge in any reasonable sense to a limit function. Let, for simplicity, $h(x) = x$, and let X_1 be uniformly distributed on [0, 1]. Set $c_{ni} = 1/n + (i - 1)\delta_n/n$ for $1 \leqslant i \leqslant k$ and $c_{ni} = 1/n + (2k - i)\delta_n/n$ for $k + 1 \leqslant i \leqslant 2k$, $k = k(n) = [n^{1/2+\varepsilon}]$, and $\delta_n = n^{-1/2-\varepsilon}$. Thus, the function $c_n(t)$ is defined on the interval $[0, 2k/n]$. On the remaining part of [0, 1], we extend $c_n(t)$ periodically with period $2k/n$ (by the parallel shift of the "tooth" constructed). Note that $1 \leqslant c_n(t) \leqslant 2$. From this we easily get that $|Y_{n1}| \leqslant 1$ and $1/12 \leqslant \sigma_n^2 \leqslant 1/3$. Then all conditions of Theorem 5 are satisfied and, consequently, relation (39) holds.

Proof of Theorem 5. Integrating by parts, we get

$$\begin{aligned} L_n - \mu_n &= \int_{\mathbb{R}} h(t)\,\mathrm{d}\bigl\{\varphi_n(F_n(t)) - \varphi_n(F(t))\bigr\} \\ &= \int_{\mathbb{R}} \bigl\{\varphi_n(F(t)) - \varphi_n(F_n(t))\bigr\}\mathrm{d}h(t). \end{aligned} \tag{40}$$

One easily sees that the last integral is well defined because of (38). We further have

$$\varphi_n(F(t)) - \varphi_n(F_n(t)) = c_n(F(t))(F(t) - F_n(t)) + R_n(t), \tag{41}$$

where

$$R_n(t) = \int_{F_n(t)}^{F(t)} \{c_n(x) - c_n(F(t))\}\,\mathrm{d}x$$

(here the integration bounds are not ordered). Let us show that

$$|R_n(t)| \leqslant n^2 \tilde{c}_n \big(F_n(t) - F(t)\big)^2. \tag{42}$$

Note that $R_n(t) = 0$ if $|F_n(t) - F(t)| \leqslant 1/n$. Let now $(k-1)/n < F(t) \leqslant k/n$, $F_n(t) = m/n$, where $|m-k| \geqslant 1$. If $m \geqslant k+1$, then

$$\begin{aligned} |R_n(t)| &\leqslant \sum_{j=k+1}^{m} |c_{nj} - c_{nk}| \leqslant \sum_{j=k+1}^{m} \sum_{i=k}^{j-1} |c_{n,i+1} - c_{ni}| \leqslant \tilde{c}_n (m-k)^2 \\ &\leqslant n^2 \tilde{c}_n (F_n(t) - F(t))^2. \end{aligned}$$

The case $m \leqslant k-1$ is similarly studied.

Consider independent identically distributed random variables

$$Y_{ni} = \int_{\mathbb{R}} c_n(F(t))\big(F(t) - \mathbb{I}\{X_i < t\}\big)\,\mathrm{d}h(t), \quad i = 1, \ldots, n.$$

It is obvious that $\mathbb{E}Y_{n1} = 0$ and

$$\begin{aligned} \mathbb{E}Y_{n1}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} & c_n(F(x))c_n(F(y))\mathbb{E}\big(F(x) - \mathbb{I}\{X_i < x\}\big) \\ & \times \big(F(y) - \mathbb{I}\{X_i < y\}\big)\,\mathrm{d}h(x)\,\mathrm{d}h(y) = \sigma_n^2. \end{aligned}$$

Set $S_n = \sum_{i=1}^{n} Y_{ni}$. From (40) and (41) we get the representation

$$\frac{\sqrt{n}(L_n - \mu_n)}{\sigma_n} = \frac{S_n}{\sigma_n \sqrt{n}} + \frac{\sqrt{n}}{\sigma_n} \int_{\mathbb{R}} R_n(t)\,\mathrm{d}h(t). \tag{43}$$

Since relation (36) actually is the Lindeberg condition for the sequence of series of random variables $\{Y_{ni};\ i \leqslant n\}$, we have that

$$\sigma_n^{-1} n^{-1/2} S_n \xrightarrow{\mathrm{d}} \mathbb{N}(0, 1) \quad \text{as} \quad n \to \infty.$$

Denote by r_n the second summand on the right-hand side of (43). To complete the proof of the theorem, it remains to show that $r_n \underset{\mathrm{p}}{\longrightarrow} 0$ as $n \to \infty$. For this, in turn, it suffices to show that $\mathbb{E}|r_n| \to 0$ as $n \to \infty$. From (42) we get the estimate $\mathbb{E}|R_n(t)| \leqslant n\tilde{c}_n F(t)(1 - F(t))$. From this and from conditions (37) and (38) it follows that

$$\mathbb{E}|r_n| \leqslant n^{3/2} \sigma_n^{-1} \tilde{c}_n \int_{\mathbb{R}} F(t)(1 - F(t))|\,\mathrm{d}h(t)| \longrightarrow 0 \quad \text{as } n \to \infty.$$

The theorem is proved.

REFERENCES

1. A. K. Aleškevičienė, On large deviations for linear combinations of order statistics, *Liet. Matem. Rink.*, **29**(2), 212–222 (1989).
2. A. K. Aleškevičienė, Large and moderate deviations for L-statistics, *Lith. Math. J.*, **31**(2), 145–156 (1991).
3. V. Bentkus and R. Zitikis, Probabilities of large deviations for L-statistics, *Lith. Math. J.*, **30**(3), 215–222 (1990).
4. S. Bjerve, Error bounds for linear combinations of order statistics, *Ann. Statist.*, **5**(2), 357–369 (1977).
5. G. Blom, *Statistical Estimates and Transformed Beta-Variables*, Wiley, New York (1958).
6. I. S. Borisov, On the rate of convergence in the "conditional" invariance principle, *Theory Probab. Appl.*, **23**(1), 63–76 (1978).
7. I. S. Borisov, Bounds for characteristic functions of additive functionals of order statistics, *Siberian Adv. Math.*, **5**(4), 1–15 (1995).
8. I. S. Borisov and E. A. Baklanov, Moment inequalities for generalized L-statistics, *Siberian Math. J.*, **39**(3), 415–421 (1998).
9. I. S. Borisov and E. A. Baklanov, Probability inequalities for generalized L-statistics, *Siberian math. J.*, **42**(2), 217–231 (2001).
10. A. A. Borovkov, *Asymptotic Methods in Queueing Theory*, Wiley, Chichester (1984).
11. H. Callaert, M. Vandemaele, and N. Veraverbeke, A Cramér type large deviation theorem for trimmed linear combinations of order statistics, *Comm. Statist. Theory Methods*, **11**(23), 2689–2698 (1982).
12. H. Chernoff, J. L. Gastwirth, and M. V. Johns, Asymptotic distribution of linear combinations of order statistics, with applications to estimation, *Ann. Math. Statist.*, **38**(1), 52–72 (1967).
13. H. Hecker, A characterization of the asymptotic normality of linear combinations of order statistics from the uniform distribution, *Ann. Statist.*, **4**(6), 1244–1246 (1976).
14. R. Helmers, A Berry–Esséen theorem for linear combinations of order statistics, *Ann. Probab.*, **9**(2), 342–347 (1981).
15. E. H. Lloyd, Least-squares estimation of location and scale parameters using order statistics, *Biometrica*, **39**, 88–95 (1952).
16. D. M. Mason and G. R. Shorack, Necessary and sufficient conditions for asymptotic normality of L-statistics, *Ann. Probab.*, **20**(4), 1779–1803 (1992).
17. R. Norvaiša, The central limit theorem for L-statistics, *Studia Sci. Math. Hungar.*, **33**, 209–238 (1997).
18. R. Norvaiša and R. Zitikis, Asymptotic behaviour of linear combinations of functions of order statistics, *J. Statist. Plann. Inference*, **28**, 305–317 (1991).
19. I. F. Pinelis and A. I. Sakhanenko, Remarks on inequalities for large deviation probabilities, *Theory Probab. Appl.*, **30**(1), 143–148 (1986).
20. R. J. Serfling, *Approximation Theorems of Mathematical Statistics*, Wiley, New York (1980).
21. R. J. Serfling, Generalized L-, M-, and R-statistics, *Ann. Probab.*, **12**(1), 76–86 (1984).
22. S. M. Stigler, Linear functions of order statistics, *Ann. Math. Statist.*, **40**(3), 770–788 (1969).
23. S. M. Stigler, Linear functions of order statistics with smooth weight functions, *Ann. Statist.*, **2**(4), 676–693 (1974).
24. M. Vandemaele and N. Veraverbeke, Cramér type large deviations for linear combinations of order statistics, *Ann. Probab.*, **10**(2), 423–434 (1982).
25. S. S. Wilks, *Mathematical Statistics*, Wiley, New York, London (1962).
26. W. Feller, *An Introduction to Probability Theory and its Applications*, Wiley, New York (1984).
27. R. Zitikis, Smoothness of distribution function of $\mathcal{FL}$-statistic. I, *Lith. Math. J.*, **30**(2), 97–106 (1990).
28. R. Zitikis, Smoothness of distribution function of $\mathcal{FL}$-statistic. II, *Lith. Math. J.*, **30**(3), 231–240 (1990).

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