We consider the last-passage percolation model on oriented complete graphs with Gumbel weights. This model is defined through the following recursive equation: $W_0 = 0$ and

$$\forall n \geq 0, W_{n+1}^{(a)} = \max_{j \leq n}(W_j^{a)} + G_j^{(n+1)} - a), \tag{0.1}$$

with $(G_j^{(n)}, n \geq 0, j \geq 0)$ i.i.d. standard Gumbel random variables.

By standard properties of Gumbel random variables, observe that one can rewrite the above formula as

$$\forall n \geq 0, W_{n+1}^{(a)} = a + \log\left(\sum_{j=0}^{n} e^{W_j^{(a)}}\right) + G_{n+1},$$

where $(G_n, n \geq 1)$ are i.i.d. Gumbel random variables.

In particular, setting $S_n^{(a)} = \log\left(\sum_{j=0}^{n} e^{W_j^{(a)}}\right)$, we have

$$\begin{aligned} S_{n+1}^{(a)} &= \log\left(e^{W_{n+1}^{(a)}} + \sum_{j=1}^{n} e^{W_j^{(n)}}\right) \\ &= \log\left(e^{S_n^{(a)}} + e^{G_{n+1}+a} e^{S_n^{(a)}}\right) \\ &= S_n^{(a)} + \log\left(1 + e^{G_{n+1}+a}\right). \end{aligned}$$

Therefore, $(S_n^{(a)}, n \geq 0)$ is a random walk. Using that Gumbel variables are L^1, we immediately deduce the following formula for the weight growth of the last passage percolation in that case:

$$\lim_{n\to\infty} \frac{1}{n} W_n^{(a)} = \mathbb{E}\left(\log\left(1 + e^{G+a}\right)\right) = a + \gamma + e^{e^a} \mathrm{Ei}(1, e^a) =: v_a. \tag{0.2}$$

Note that we have

$$v_a = e^a(-a + \gamma - 1)(1 + o(1)) \text{ as } a \to \infty.$$

It would be interesting to compare it to the Barak-Erdős graph. Using a simple comparison setting

$$(G + a) \geq \varepsilon \mathbf{1}_{\{G+a>\varepsilon\}} - \infty \mathbf{1}_{\{G+a<\varepsilon\}},$$

we have

$$\begin{aligned} v_a &\geq \varepsilon C(\mathbb{P}(G + a > \varepsilon)) = \varepsilon C(1 - e^{-e^{a-\varepsilon}}) \\ &\approx \varepsilon C(e^{a-\varepsilon}) \approx \varepsilon e e^{a-\varepsilon}\left(1 - c/(a-\varepsilon)^2\right). \end{aligned}$$

We can also take interest in the path being the rightmost one at time n. Observe that

$$\mathbb{P}\left(W_{n+1} = W_j + G_j^{(n+1)} \middle| \mathcal{F}_n\right) = \frac{e^{W_j^{(a)}}}{\sum_{i=1}^{n} e^{W_i^{(a)}}},$$

therefore we can construct an infinite path as follows: starts with the random walk $(-S_n, n \geq 0)$ and then define recursively the value of w_n by setting

$$\mathbb{P}(w_{n+1} = j + k | S, w_n = k) = e^{G_{j+k} - S_k}.$$

It consists of a random walk, whose step distribution can be computed explicitly.